1. Introduction and Basic Definitions

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CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID

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ABSTRACT. The centre of the braid group B_n is well-known to be infinite cyclic and generated by a twist braid. In this paper we consider the centraliser of certain important subgroups, and in particular we characterise the elements of B_n which commute with one of the usual generators σ_j. This characterisation is generalised to the monoid of singular braids SB_n, recently introduced (independently) by J. Baez and J. Birman. We determine the singular braids which commute with σ_j, or with a singular generator τ_j; in fact we show these submonoids are the same.

We establish that the centraliser in B_n of σ_j is isomorphic to the cartesian product of two groups: the group of $(n-1)$-braids whose permutations stabilise j and the group of integers. More generally, we show that the centraliser of the naturally-included braid subgroup $B_r \subset B_n$ likewise splits as a direct product, and we give an explicit presentation for this centraliser. We also describe the centralisers of $SB_r \subset SB_n$.

As another application we consider a conjecture of J. Birman regarding the injectivity of a map, related to Vassiliev theory, $\eta: SB_n \to \mathbf{Z}B_n$ from the singular braid monoid to the group ring of the braid group. We see that the question is related to the centraliser problem and prove the injectivity of η for braids with up to two singularities.

1. INTRODUCTION AND BASIC DEFINITIONS

The braid group B_n, for an integer $n \geqslant 2$, may be considered abstractly as the group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_j \sigma_k = \sigma_k \sigma_j \qquad \text{if} \quad |j-k| > 1,$$
$$\sigma_j \sigma_k \sigma_j = \sigma_k \sigma_j \sigma_k \qquad \text{if} \quad |j-k| = 1\,.$$

There are equivalent geometric descriptions of braids as strings in space, as automorphisms of a free group F_n, as the fundamental group of a configuration space, or as homeomorphisms of an n-punctured plane (see below), which explains the importance of the braid groups in many

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disciplines. The originator of braid theory, Emil Artin, posed several (at the time) "unsolved problems" in [Art2], including:

"With what braids is a given braid commutative?"

"Decide for any two given braids whether they can be transformed into each other by an inner automorphism of the group."

The present paper is concerned primarily with the first question: finding the centraliser of a given braid. Although an algorithm exists, as we'll describe shortly, this problem can still be said to be open in general. However, we consider the most basic special case and characterise, in a simple geometric way, the set of all braids which commute with one of the generators σ_j.

The latter question — the conjugacy problem — was settled in principle by Garside [Gar], who gave a finite procedure to decide if two given braids are conjugate. The present work also contributes to this question, in that we determine exactly which inner automorphisms will take one of the standard generators to another.

G. Burde [Bur] has computed the centralisers of certain special kinds of braids: those which are "j-pure" as defined by Artin, meaning a pure braid (see our discussion below) for which all strings except the j^{th} are horizontal straight lines. Burde's point of view (like ours, which was developed independently) is partly algebraic and partly geometric.

For an arbitrarily given element $\alpha \in B_n$, there is an algorithm to find a finite set of generators for its centraliser, as shown by G. Makanin [Mak]. This result was extended by G. Gurzo [Gur1] to the centraliser of any finite set of elements of B_n, who showed that the generators can be taken to be positive braids (no negative exponents). The methods of Makanin and Gurzo are algebraic and combinatorial. They rely heavily on techniques pioneered by Garside; transferring the problem to the monoid of positive braids, and thus making its solution a finite search. Their method sheds little light on the actual structure of the centraliser. However, in a later paper [Gur2], Gurzo extended the work to explicitly compute generating sets for centralisers of various special types of braids, including the σ_j and their powers. As an application, she discovered that the centraliser of any nonzero power σ_j^m is independent of m.

In fact more can be said, in general, of centralisers of finite sets in B_n; they are *biautomatic.* Thurston proved in [ECHLPT] that B_n is biautomatic (see also [Char1], [Char2]), and Gersten-Short [GS] have shown that centralisers of finite sets in biautomatic groups are themselves biautomatic.

Following some preliminaries, our goal in Sections 2 and 3 will be to characterise the centraliser of σ_j using the geometric viewpoint, exploiting the action of B_n on (classes of) arcs in the complex plane. More generally, we identify all solutions β to $\sigma_j \beta = \beta \sigma_k$ by a natural criterion involving braids as geometrical objects — having what we call a "(j, k)-band." Using this criterion, we recover Gurzo's result that the centraliser of σ_j^m is independent of $m \neq 0$. It also gives an alternative proof to the old result [Chow] that the centre of B_n is infinite cyclic. In Section 4 we use our results to describe the structure of the centraliser of B_r in B_n, where $B_r \subset B_n$ is the usual inclusion, $r \leqslant n$, and we give an explicit presentation of this centraliser (our generators are different from Gurzo's).

In Section 5 we consider an extension of B_n to the singular braid monoid SB_n recently introduced by Birman [Bir2] and Baez [Bae] to study Vassiliev theory. We show that the centraliser of a basic singular generator τ_j in SB_n coincides with the centraliser of σ_j. Moreover, the solutions β to $\tau_j \beta = \beta \tau_k$ are shown to be exactly those β which have a (possibly singular) (j, k)-band.

Our results are used in Section 6 to study a question raised by Birman regarding injectivity of the Baez-Birman-Vassiliev map η from SB_n into the group ring $\mathbf{Z}B_n$. Finally, in Section 7 we generalise the "Band Theorem" (2.2) to the context of singular braids, and consider the centralisers of σ_j, τ_j and SB_r in SB_n.

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Geometric braids. Let $\mathbf{C}$ denote the complex plane, $\{1, \ldots, n\}$ the first n integral points on the positive real axis and $\mathbf{I} = [0, 1]$ the unit interval. We consider an n-braid β to be a collection of n disjoint strings $\beta \subset \mathbf{C} \times \mathbf{I} = \{(z, t)\}$ such that the j-th string runs, monotonically in t, from the point $(j, 0)$ to some point $(k, 1), j, k \in \{1, \ldots, n\}$. An isotopy in this context is a deformation through braids (with fixed ends), and isotopic braids are considered equivalent. We write $j = \beta * k$ or, equivalently $j * \beta = k$, so that braids can act either on the right or left as permutations of $\{1, \ldots, n\}$. A *pure* braid is one whose permutation is the identity. We will picture braids horizontally rather than vertically, so that multiplication of braids is by concatenation from left to right, just as written algebraically. The (equivalence classes of) braids with this multiplication form the group B_n described algebraically above.

Basic references for braid theory are [Art1], [Art2] and [Bir1]; [BZ] and [Han] also contain good accounts and [Bir2] is an up-to-date discussion including singular braid theory. As noted by Artin, one can also regard a braid as corresponding to a homeomorphism of $\mathbf{C}$ onto itself, with compact support and setwise preserving $\{1, \ldots, n\}$. More precisely, a braid corresponds to a *mapping class* in which homeomorphisms of $\mathbf{C}$ are considered equivalent if one can be obtained from the other by a (compactly supported) isotopy in which the points $\{1, \ldots, n\}$ are held fixed. Thus equivalence classes of braids are in one-to-one correspondance with such mapping classes. As depicted in Figure 1 (see also Figure 6), the braid σ_j corresponds to the class of a homeomorphism which is a local (right-hand) twist of the plane interchanging the points j and $j+1$, and supported on a neighbourhood of the interval $[j, j+1]$.

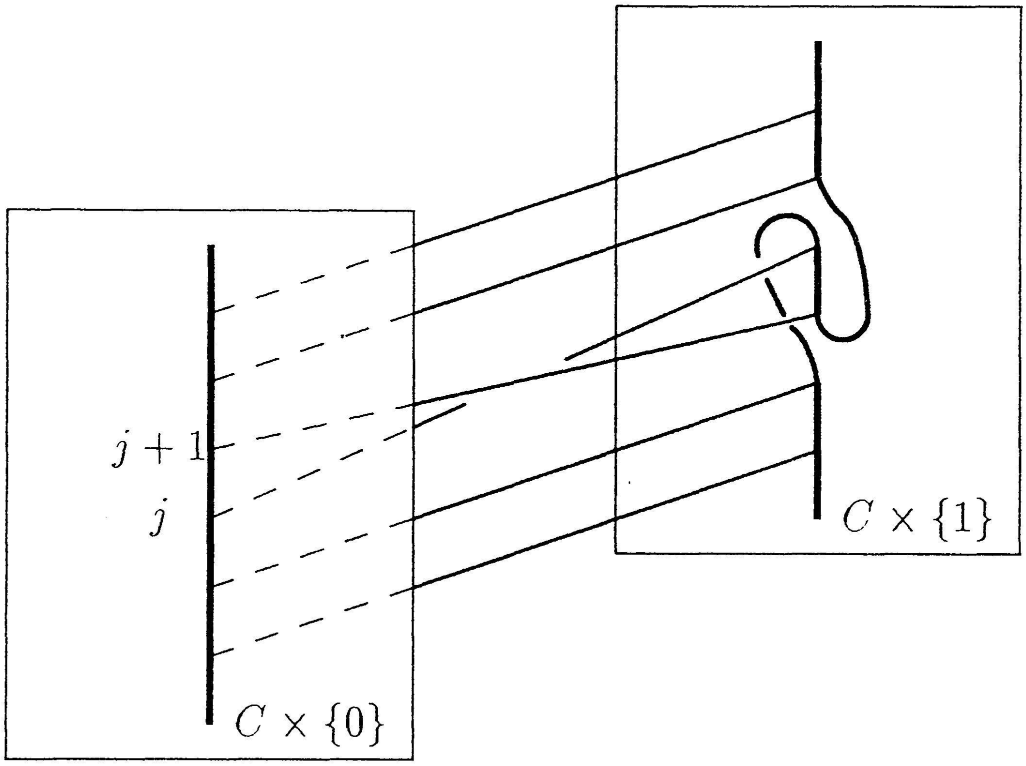

FIGURE 1
The homeomorphism associated with a generator σ_j

The inverse correspondence is as follows: suppose one has a homeomorphism of $\mathbf{C}$ which is compactly supported and fixes $\{1, \ldots, n\}$ setwise. This homeomorphism is isotopic to the identity, but the points $\{1, \ldots, n\}$ may move during the isotopy. The track of these points, in $\mathbf{C} \times \mathbf{I}$, through the isotopy, gives the geometric braid corresponding to (the class of) the given homeomorphism.

The product of braids corresponds to composition of homeomorphisms of $\mathbf{C}$. One can have the braid group act on $\mathbf{C}$ either on the left or right — both conventions appear in the literature. It is convenient for us, in fact, to adopt both conventions, extending the above notation for permutations so that

$$*\beta : \mathbf{C} \to \mathbf{C} \text{ corresponds to a mapping } \mathbf{C} \times 0 \to \mathbf{C} \times 1 ,$$

and defines an action on the right, whereas

$$\beta * : \mathbf{C} \to \mathbf{C} \text{ corresponds to a mapping } \mathbf{C} \times 1 \to \mathbf{C} \times 0$$

and operates on the left. Thus, for any subset X of $\mathbf{C}$, and braids α, β:

$$X * (\alpha\beta) = (X * \alpha) * \beta,$$
$$(\alpha\beta) * X = \alpha * (\beta * X), .$$
$$X * \beta = \beta^{-1} * X$$

This action extends the permutation action of B_n as discussed earlier. (We note that our depiction of generators σ_j disagrees with that of some earlier authors, but is in keeping with recent practice, so that σ_j corresponds to a "positive" oriented crossing; a right-handed twist instead of left.)

PROPER ARCS AND RIBBONS. An important rôle will be played by the set of arcs in $\mathbf{C}$ which are *proper rel* $\{1, \ldots, n\}$, by which we mean that their endpoints are in the set $\{1, \ldots, n\}$ and their interiors are disjoint from that set. Such an arc from (say) j to k is called a (j, k)-arc. We consider two (j, k)-arcs equivalent if they are connected by a continuous family of proper arcs; in other words, isotopic. Unless otherwise stated, we do *not* distinguish a (j, k)-arc A from its *reverse*, the oppositely oriented $\bar{A}$, which is a (k, j)-arc. Use the notation:

$\mathbf{A}_n$ = the set of proper arcs in $\mathbf{C}$, modulo isotopy fixing $\{1, \ldots, n\}$.

It is clear from the above discussion that the braid group also acts naturally on $\mathbf{A}_n$, and we adopt the same symbols $\beta *$ and $* \beta$ for the left and right actions.

By a *ribbon* we will mean an embedding

$$R : \mathbf{I} \times \mathbf{I} \to \mathbf{C} \times \mathbf{I}$$

such that $R(s, t) \in \mathbf{C} \times t$. Suppose one has a braid β and a (j, k)-arc A in $\mathbf{C} \times 0$. Then the isotopy corresponding to β moves A through a ribbon which is *proper* for β, meaning $R(0, t)$ and $R(1, t)$ trace out two strands of the braid, while the rest of the ribbon is disjoint from β. The left end of the ribbon is A and the right end is $A * \beta$.

1.1 PROPOSITION. *Let β be an n-braid and A and B be proper arcs for $\{1, \ldots, n\}$. Then $A * \beta = B$ if and only if there is a proper ribbon for β connecting $A \subset \mathbf{C} \times 0$ to $B \subset \mathbf{C} \times 1$.*

Proof. We have already argued that $A * \beta = B$ implies the existence of the ribbon. On the other hand, suppose there is a ribbon R from A to B proper for β. Then by reflection of the ribbon from A to $A * \beta$ and concatenation with R, one has a ribbon from $A * \beta$ to B along $\beta^{-1}\beta$. But $\beta^{-1}\beta$ can be moved by level-preserving isotopy to the trivial braid $\{1, \ldots, n\} \times \mathbf{I}$, and then the image of the ribbon provides an isotopy from $A * \beta$ to B fixing $\{1, \ldots, n\}$. $\square$

2. Commutation and stabilisers

The theme of this paper is to reflect algebraic properties of a braid in the geometry of ribbons and the action of B_n on $\mathbf{A}_n$.

Consider an n-braid β which is constructed from an $(n-1)$-braid by running a narrow ribbon along the j^{th} string, with the ends of the ribbon being straight line segments on the real line, as pictured in Figure 2. The ribbon may be twisted arbitrarily. Let β consist of the two edges of the ribbon, together with the other strands of the $(n-1)$-braid (those of index greater than j need to be renumbered and have their ends shifted, of course.) Premultiplying β by σ_j corresponds to putting a twist in the left end of the ribbon, and the ribbon can be used to convey that twist through β until it emerges on the right, and we have the equation: $\sigma_j \beta = \beta \sigma_k$.

In the special case of $j = k$ we have constructed a class of braids which commutes with the generator σ_j. In fact, if β is any braid for which $[j, j+1] * \beta = [j, j+1]$, it can be isotoped, with fixed endpoints, into one with such parallel strands. Just slide the strands near each other along the ribbon, but taper to the identity to keep the ends fixed.

Definition. We say that β *has a (j, k)-band* if there exists a ribbon (the band) proper for β and connecting $[j, j+1] \times 0$ to $[k, k+1] \times 1$.

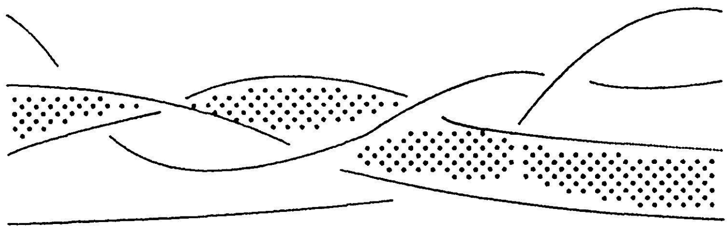

Figure 2
A braid with a (2.1)-band