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OVER TORSION-FREE GROUPS  
**Autor:** Fenn, Roger / Rourke, Colin  
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notation. Let  $r_i$  be the road common to the boundary of  $D$  and the country with stopping car  $\kappa_i$ . Let the end junctions of  $r_i$  be  $v_i$  and  $v_{i+1}$  in anticlockwise order  $i = 1, 2, \dots$ . Suppose  $\kappa_i$  is on the road  $r_i$ . Then by hypothesis the roads  $r_{i+1}$  and  $r_{i-1}$  are free of the cars  $\kappa_{i+1}$  and  $\kappa_{i-1}$  respectively (see figure 3).

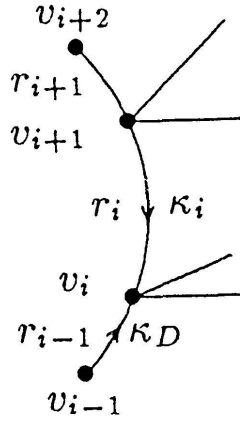


FIGURE 3

The motion of car  $\kappa_D$

As  $\kappa_i$  traverses  $r_i$  from  $v_{i+1}$  to  $v_i$  let  $\kappa_D$  traverse  $r_{i-1}$  from  $v_{i-1}$  to  $v_i$ . Let the cars meet at  $v_i$  at time  $t$ . This will not be a complete crash since  $\kappa_{i-1}$  is missing. Again by hypothesis  $\kappa_{i+1}$  will not be at  $v_{i+2}$  at time  $t$ . Let  $r$  be largest such that  $\kappa_{i+r}$  is not at  $v_{i+r+1}$  at time  $t$ . Then  $\kappa_D$  has enough time to arrive at  $v_{i+r+1}$  just as  $\kappa_{i+r}$  does. If there is no such  $r$  then let  $\kappa_{i+r}$  be the next car to arrive at  $D$  and let  $\kappa_D$  go once round the entire boundary and arrive at  $v_{i+r+1}$  just as  $\kappa_{i+r}$  does. Keep repeating this strategem to define the motion of  $\kappa_D$ .

Now we are in a situation corresponding to the first crash theorem and the result is proved.  $\square$

### 3. TWO TRANSVERSALITY LEMMAS

In this section we use transversality (cf. [BRS, F]) to prove the existence of diagrams of van-Kampen type for the two situations that we shall meet in the applications to group theory of the crash theorems (in sections 4, 5 and 6). These lemmas need to be stated very carefully and a failure to do so is one of the major weaknesses in Klyachko's version. The lemmas use the idea of a *corner* of a 2-cell in a cell subdivision  $K$  of the 2-sphere. This can be regarded as the (oriented) angle formed by the two adjacent edges meeting at a 0-cell in the boundary of the 2-cell. If all the corners of a 2-cell are labelled by elements of a group, then a word can be read around the 2-cell

boundary by composing these elements either unchanged or inverted according as the orientation of the corner agrees or disagrees with that of the 2-cell boundary. Similarly if all the corners at a 0-cell are labelled then a word can be read around that 0-cell. We shall always orient corners *clockwise*, thus if the above words are read *clockwise* for 0-cells and *anticlockwise* for 2-cells, then no inversion is necessary (see figure 4).



FIGURE 4

Multiplying the corner labels to get  $g_1 g_2 \cdots g_n$  for a 0-cell and a 2-cell

Let  $w \in A * B$  be an element in the free product of two groups. We shall only be interested in  $w$  up to cyclic reordering and thus (cyclically reordering if necessary) we can assume that either  $w = 1$  or  $w \in A \cup B - 1$  or  $w$  is written uniquely as  $a_1 b_1 a_2 b_2 \cdots a_n b_n$  where  $a_i$  and  $b_i$  are non-trivial elements of  $A, B$  alternately. These non-trivial elements  $a_i, b_i$  are then called the (cyclic) *factors* of  $w$ .

LEMMA 3.1. *Let  $A, B$  be two groups and let  $N = \langle\langle W \rangle\rangle$  be the normal closure in  $A * B$  of some subset of elements  $W \subset A * B$ . Suppose  $N \cap A \neq \{1\}$ . Then there is a cell subdivision of the sphere  $S^2$  such that each corner of each 2-cell is labelled by an element of  $A \cup B$  with the following properties.*

1. *The corner labels of a 2-cell are the cyclic factors (in anticlockwise order and up to cyclic rotation) of some  $w$  or  $w^{-1}$  where  $w \in W$ .*
2. *The corner labels at a 0-cell are either all in  $A$  or all in  $B$ .*
3. *The (clockwise) product of the corner labels at a 0-cell is 1 (in  $A$  or  $B$ ) except for one special 0-cell where the product is a non-trivial element of  $A \cup B$ .*

*Proof.* Let  $K_A, K_B$  be two disjoint 2-dimensional complexes such that  $\pi_1(K_A, *_A) = A$  and  $\pi_1(K_B, *_B) = B$ . Join the base points  $*_A$  and  $*_B$  by an arc  $\alpha$  with central point  $*$ . Let  $K = K_A \cup \alpha \cup K_B$ . Then  $\pi_1(K, *)$

$\cong A * B$ . Attach 2-cells  $\sigma_w$  to  $K$  by the words  $w \in W$  to form the complex  $L$ . If  $a \in N \cap A - \{1\}$  there is a map  $f: D^2, S^1 \rightarrow L, K$  from the 2-disc to  $L$  such that the restriction  $f|S^1$  to the boundary represents  $a$ . Make the map  $f$  transverse to the centres of the 2-cells  $\sigma_w$ . It follows that the inverse images of small neighbourhoods of these centres is a collection of disjoint discs  $D_1, \dots, D_m$  in the interior of  $D^2$ . By a radial expansion of  $f$  on these discs we may assume that each image is the whole of one of the  $\sigma_w$ . It follows that the punctured disc  $P = D^2 - \overline{D_1 \cup \dots \cup D_m}$  is mapped by  $f$  to  $K$ . Make  $f|P$  transverse to  $*$ . Then  $f^{-1}*$  is a 1-manifold  $Z$  properly embedded in  $P$ . By a radial expansion along  $\alpha$  we can assume that  $Z$  has a neighbourhood  $N$  which is a normal  $I$ -bundle and where each fibre is mapped by  $f$  to  $\alpha$ . The complementary space  $P - N$  is divided into connected regions which are mapped by  $f$  to  $K_A$  or  $K_B$ . On crossing  $N$  one passes from one kind of region to the other.

We now simplify the subset  $D_1 \cup \dots \cup D_m \cup N$  of  $D^2$  as follows. Suppose  $N$  contains an annulus component  $\mathcal{A}$  in the interior of  $P$ . Let  $D'$  denote the interior disc of  $D^2$  which bounds the interior boundary component of the annulus. Then  $D' \cup \mathcal{A}$  is a sub disc of  $D^2$  whose boundary gets mapped to a base point by  $f$ . We can then shrink it to a point, redefine  $f$  and simplify the situation. Having eliminated all annuli,  $D_1 \cup \dots \cup D_m \cup N$  will look like a thickened graph in  $D^2$  with the discs  $D_i$  corresponding to thickened vertices and the components of  $N$  to thickened edges. Our next task is to make this graph connected. If not choose an innermost component  $C$ . Draw a simple loop around  $C$  separating it from the rest of  $D_1 \cup \dots \cup D_m \cup N$ . This loop will represent (up to conjugacy) an element of  $A \cup B$ . If this element is trivial we can shrink the disc it bounds as above and simplify the situation. If not we replace  $D_1 \cup \dots \cup D_m \cup N$  by  $C$ . Note that the boundary curve may now represent a non trivial element of  $B$  instead of  $A$ .

Attach a 2-cell (outside) to the boundary of  $D^2$  and label the centre of this outside cell  $\infty$ . The 2-disc has now become a 2-sphere. In this situation consider the dual graph  $\Gamma$ . This has a vertex in each region and an edge joining neighbouring regions separated by a component of  $N$ . For the outer region take the vertex to be  $\infty$ . Then  $\Gamma$  and its complementary regions define a cell subdivision  $K$  of the 2-sphere. Each vertex is either in an  $A$  region or a  $B$  region and the corners can be correspondingly labelled by elements of  $A$  or  $B$  as follows. Every 2-cell of  $K$  contains a unique subdisc  $D_i$ . Opposite a corner is an edge of  $D_i$  labelled by an element of  $A$  or  $B$ . Take this to be the labelling of the corner. By moving anticlockwise around the

boundary of a 2-cell of  $K$  the corner labellings spell out a cyclic rotation of some  $w_i$  or  $w_i^{-1}$ . By moving clockwise around a 0-cell of  $K$  the corner labellings spell out the trivial element (of  $A$  or  $B$ ) except for  $\infty$  which spells out a non-trivial element of  $A$  or  $B$ .  $\square$

NOTE. It may not be possible to specify that the non-trivial element lies in  $A$  as this simple example shows. Let  $A = \langle a \rangle$ ,  $B = \langle b \rangle$  be two infinite cyclic groups generated by  $a, b$  respectively. Let the words of the attaching 2-cells be  $ab^{-1}, b$ . In this case the 2-cells of the required subdivision have either two corners (those modelled on  $ab^{-1}$ ) or one corner (modelled on  $b$ ) and the only possible subdivision of the 2-sphere satisfying lemma 3.1 is the trivial one with single vertex labelled  $b$ . This is a place where Klyachko's version is definitely wrong (rather than badly stated).

Let  $w \in G * \langle t \rangle$  be an element of the free product of a group  $G$  with the infinite cyclic group  $\langle t \rangle$ . Then  $w$  can be written uniquely (up to cyclic rotation) in the form  $w = g_1 t^{\varepsilon_1} g_2 \cdots t^{\varepsilon_n}$  where each  $g_i \in G$ , each  $\varepsilon_i = \pm 1$  and  $g_i$  can only be 1 if it has neighbouring  $t$ 's (in cyclic order) with the same exponent. We call  $g_1, \dots, g_n$  the *coefficients* of  $w$ .

The following lemma is proved in [H<sub>1</sub>]. It is closely related to "pictures" [R<sub>1</sub>, R<sub>2</sub>, Sh].

LEMMA 3.2. *Let  $G$  be a group and consider the free product  $G * \langle t \rangle$  of  $G$  with an infinite cyclic group (generator  $t$ ). Let  $N = \langle\langle W \rangle\rangle$  be the normal closure in  $G * \langle t \rangle$  of some subset of elements  $W \subset G * \langle t \rangle$ . Suppose  $N \cap G \neq \{1\}$  then there is a cell subdivision  $K$  of the 2-sphere such that*

- a) *the 1-cells of  $K$  are oriented,*
- b) *the corners (all oriented clockwise) are labelled by coefficients of elements of  $W$ ,*
- c) *the clockwise product of the corner labelling around any 0-cell is 1 except for one vertex where it is non trivial,*
- d) *the corner labels of any 2-cell (in anticlockwise order) are the coefficients of  $w$  or  $w^{-1}$  for some  $w \in W$  (up to cyclic rotation) with the property that, if on passing from one corner to an adjacent corner the element  $t$  or  $t^{-1}$  is inserted according to whether the intervening edge is oriented in the same or opposite direction, then the whole of  $w$  or  $w^{-1}$  is recovered.*

*Proof.* The proof is very similar to 3.1. Let  $K_G$  be a 2-dimensional complex such that  $\pi_1(K_G, *_G) = G$ . Adjoin an oriented 1-cell  $\gamma$  to the base point  $*_G$  to form a 2-dimensional complex  $K = K_G \vee S^1$  with  $\pi_1 K = G * \langle t \rangle$ . Attach 2-cells to  $K$  by the words  $w \in W$  to form  $L$ . Since  $N \cap G \neq \{1\}$  there is a non contractable loop in  $K_G$  represented by a map  $f: S^1, 1 \rightarrow K_G, *_G$  which can be extended to a map  $f: D^2 \rightarrow L$ .

We now proceed as in the proof of lemma 3.1 with the rôle of  $*$  played by a point  $p$  in the interior of  $\gamma$ . We construct a graph whose (thickened) vertices are the inverse image of the 2-cells and whose edges are the inverse image of  $p$ . By making similar simplifications and passing to an innermost component, as before, we may assume that this graph is connected. Replace  $D^2$  by a sphere as before. The dual subdivision now defines  $K$ . The orientation of the 1-cells is determined by the orientation of  $\gamma$  and it only remains to observe that these oriented edges correspond to the new generator  $t$ .  $\square$

#### 4. APPLICATION TO THE KERVAIRE PROBLEM

In this section we give Klyachko's application of the crash theorems to prove theorem 1.1 in the case in which exponent sum of  $t$  in the word  $w$  is 1. As remarked in the introduction this implies the Kervaire conjecture for torsion-free groups.

We say that a system of equations  $\{w(t) = 1 \mid w \in W\}$  in the variable  $t$ , with coefficients in a group  $G$ , has a *solution over  $G$*  if there is a group  $\tilde{G}$  containing  $G$  as a subgroup and an element  $x \in \tilde{G}$  such that the relations  $\{w(x) = 1 \mid w \in W\}$  are satisfied in  $\tilde{G}$ . It is clear that this is equivalent to the natural map

$$G \rightarrow \frac{G * \langle t \rangle}{\langle\langle W \rangle\rangle}$$

being injective, where  $\langle\langle W \rangle\rangle$  denotes the normal closure of  $W$  in  $G * \langle t \rangle$ .

Now let  $H$  be a subgroup of  $G$  and let  $g \in G$ . We say that  $g$  is *free relative to  $H$*  if the subgroup  $\langle g, H \rangle$  of  $G$  generated by  $g$  and  $H$  is naturally the free product  $\langle g \rangle * H$  of an infinite cyclic group  $\langle g \rangle$  with  $H$ .

We shall apply the crash theorem with stops to prove theorem 4.1 (below) and then use an algebraic trick to deduce the case  $\text{ex}(w) = 1$  of theorem 1.1.

If  $g, h$  are elements of a group let  $g^h$  denote  $h^{-1}gh$ .