1. Introduction

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In [Kl] Klyachko gives a proof of the Kervaire conjecture in the situation where the groups involved are torsion-free. Unfortunately the paper suffers from defects. Apart from a deficiency in the use of English many theorems are ill-explained and even wrong unless interpreted exactly. The blame for this situation must be attributed to the editors of the journal. In this paper our aim is to state all his results carefully and explain the proofs in detail. Moreover using his methods we shall generalise his main theorem.

We consider the free product $G * \langle t \rangle$ of G with an infinite cyclic group generated by t. Let $w \in G * \langle t \rangle$. Then w has a unique expression of the form $g_0 t^{q_1} g_1 t^{q_2} \ldots g_{n-1} t^{q_n} g_n$ where $g_i \in G$ are non-trivial for $0 < i < n$ and q_i are non-zero integers for each i. We call $\sum_{i=1}^{n} q_i$ the *exponent sum* of t in w, denoted $\mathrm{ex}(w)$, and $\sum_{i=1}^{n} |q_i|$ the *t-length* of w. The unreduced word $t^{q_1} t^{q_2} \ldots t^{q_n}$ obtained by deleting the elements of G from w is called the *t-shape* of w.

Let $\langle\langle w \rangle\rangle$ denote the normal closure of w in $G * \langle t \rangle$.

The two main problems with which we shall be concerned are the Kervaire problem and the adjunction problem, which we now state.

The Kervaire problem. *Suppose G is a non-trivial group. Is it possible for $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ to be trivial?*

The negative answer to the Kervaire problem is known as the *Kervaire conjecture* after a conversation between M. Kervaire and G. Baumslag c. 1963 [Ke, p. 117, MKS p. 403] and it has been proved for large classes of groups, for example compact topological groups, locally residually finite groups [GR, Ro], locally indicable groups [H_2, Sh]. In general however the problem is still open.

Now think of $w = 1$ as an equation in the "variable" t with "coefficients" the elements g_i. We say that $w = 1$ *has a solution over* G if there exists a group $\tilde{G}$ with G embedded in $\tilde{G}$ and an element $x \in \tilde{G}$ such that $w(x) = 1$ where $w(x)$ is the result in $\tilde{G}$ of substituting x for t in w. It is easily seen that $w(x) = 1$ has a solution over G if and only if the natural map $G \to \frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ is injective.

The adjunction problem. *Under what circumstances does the equation $w = 1$ have a solution over G?*

Clearly it is necessary that $w \in G * \langle t \rangle - G$. However even if $w \notin G$ then G may still fail to embed in $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ as the following example shows. Let G be the cartesian product of a cyclic group of order p, generated by x, with a cyclic group of order q, generated by y, where p and q are coprime integers and let $w = xt^{-1}yt$ then $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ is infinite cyclic and G fails to embed.

Notice that this example has exponent sum zero and that the group G has torsion. If $\mathrm{ex}(w) \neq 0$ then the adjunction problem is also open in general. It is known that a solution exists in the case $\mathrm{ex}(w) = 1$ for the same classes of groups as for the Kervaire conjecture (listed above). Indeed a positive answer to the adjunction problem when $\mathrm{ex}(w) = 1$ would imply the Kervaire conjecture, since if $\mathrm{ex}(w) \neq \pm 1$ then $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ has a non-trivial cyclic quotient. It is also known that a solution exists if $|\mathrm{ex}(w)|$ is the t-length of w [L] and if $\mathrm{ex}(w) \neq 0$ and t-length $\leqslant 4$ [H_1, EH]. The problem considered here is a special case of the more general adjunction problem considered by Neumann, [N], which considers the effect of adding finitely many new generators and relators.

The main result proved here solves both problems when G is torsion-free for a large class of words, the *amenable* words. These are words whose t-shape satisfies a technical condition, and includes all words with $\mathrm{ex} = \pm 1$, for details see section 5.

THEOREM 1.1. *Let G be a torsion-free group and let $w \in G * \langle t \rangle - G$ be an amenable word. Then $w = 1$ has a solution over G.*

COROLLARY 1.2. (Klyachko: The Kervaire conjecture for torsion-free groups.) *Let G be a non-trivial torsion-free group and let w be an element of $G * \langle t \rangle - G$. Then $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ is non-trivial.*

Klyachko's paper [Kl] contains the proof of theorem 1.1 in the case in which the exponent sum of t in w is 1. (As observed above, this is the case which implies the Kervaire conjecture.) In this paper we shall take a direct path to theorem 1.1, the proof of which is given in sections 4 and 5. Klyachko proves some further results on solving equations over groups in a variety of other circumstances, and for completeness we shall give these results in section 6.

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