3. The intrinsic theory: genus zero

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3. THE INTRINSIC THEORY: GENUS ZERO

From the intrinsic point of view we start with a pair of holomorphic involutions $\tau_i: \Gamma \to \Gamma$, $i = 1, 2$, on an abstract Riemann surface Γ. The quotient spaces $\Gamma/\tau_i \equiv \Gamma_i$ have natural analytic structures [4], and τ_i is the covering involution for the branched covering $\pi_i: \Gamma \to \Gamma_i$. If

$$\tau_2 = \rho\tau_1\rho \tag{3.1}$$

for an anti-holomorphic involution ρ on Γ, then there exists an anti-biholomorphic map $\hat{\rho}: \Gamma_1 \to \Gamma_2$ with $\hat{\rho} \circ \pi_1 = \pi_2 \circ \rho$. We are mainly concerned with the case $\Gamma_1 = \Gamma_2 \subseteq \mathbf{P}_1$, although one could study real analytic curves on an arbitrary Riemann surface Γ_1. If Γ is compact, and $\Gamma_1 = \mathbf{P}_1$, then Γ is hyperelliptic. The existence of the two functionally independent 2-fold branched coverings $\pi_i: \Gamma \to \mathbf{P}_1$ forces Γ to be either an elliptic or rational curve [4]. We shall restrict to these two cases, in this paper.

In the genus zero case, $\Gamma = \mathbf{P}_1$, which we consider in this section, the holomorphic involutions are fractional linear maps. A single one $\tau(t)$ can be normalized so that its fixed points are $t = 0, \infty$, and hence has the form $\tau(t) = -t$. The theory of a pair of such involutions is still elementary, but somewhat involved, so we shall refer to [8] for some details.

For a pair of holomorphic involutions τ_1, τ_2, let the fixed-point sets be

$$FP(\tau_i) = \{p_i, q_i\}, \ i = 1, 2 \ . \tag{3.2}$$

If τ_1 and τ_2 have the same fixed-point sets, they are equal. They have a single common fixed point in the parabolic case. We first consider the general case in which the four points $\{p_1, q_1, p_2, q_2\}$ are all distinct. We may form their cross ratio,

$$\kappa = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)} \ . \tag{3.3}$$

Interchanging τ_1 and τ_2, or p_1 with q_1, or p_2 with q_2 results in (at most) the change $\kappa \mapsto 1/\kappa$. Thus, the conditions $\kappa > 0$, $\kappa < 0$, $Re\,\kappa = 0$, $\kappa\bar{\kappa} = 1$, for example, are intrinsic conditions on the pair τ_i. The first two occur when τ_1 and τ_2 are intertwined by an anti-holomorphic involution ρ. The significance of the second two conditions is still rather mysterious at this point.

The maps τ_1, τ_2 may be represented in homogeneous coordinates $(\xi, \eta) \in \mathbf{C}^2$ for $\mathbf{P}_1$ by a pair of linear involutions. As in section 2 of [8] they may chosen as follows,

$$\tau_1(\xi,\eta) = (\lambda\eta, \lambda^{-1}\xi), \quad \tau_2(\xi,\eta) = (\lambda^{-1}\eta, \lambda\xi), \tag{3.4}$$
$$\sigma(\xi,\eta) = (\mu\xi, \mu^{-1}\eta), \quad \mu = \lambda^2 .$$

In the non-homogeneous coordinate $t = \xi/\eta$,

$$\tau_1(t) = \frac{\mu}{t}, \ \tau_2(t) = \frac{1}{\mu t}, \ \sigma(t) = \mu^2 t . \tag{3.5}$$

Since

$$FP(\tau_1) = \{\lambda, -\lambda\}, FP(\tau_2) = \{\lambda^{-1}, -\lambda^{-1}\} , \tag{3.6}$$

we have

$$\kappa = \left(\frac{1-\mu}{1+\mu}\right)^2 . \tag{3.7}$$

An anti-holomorphic involution ρ of $\mathbf{P}_1$ is given by reflection in some circle, which is anti-linear in homogeneous coordinates. Thus, lemma 2.2 of [8] applies directly to give the following.

LEMMA 3.1. *The normal form for the triple* τ_1, τ_2, ρ, *with* $\tau_2\rho = \rho\tau_1$, *falls into two cases. The* τ_i *are still given by (3.4) or (3.5), while*

$$\lambda = \bar{\lambda} > 1, \quad \rho(\xi,\eta) = (\bar{\eta}, \bar{\xi}), \quad \rho(t) = 1/\bar{t} , \tag{3.8}$$

or

$$\lambda\bar{\lambda} = 1, \quad 0 < \arg\lambda < \pi/2, \quad \rho(\xi,\eta) = (\bar{\xi}, \bar{\eta}), \quad \rho(t) = \bar{t} . \tag{3.9}$$

(3.11) is the elliptic case with $\kappa > 0$. (3.12) is the hyperbolic case, where $\kappa < 0$.

Next we consider the problem of realizing the data τ_i by means of an analytic curve,

$$z = \pi_1(t), \quad \bar{w} = \pi_2(t), \quad \pi_i \circ \tau_i = \pi_i . \tag{3.10}$$

This amounts to finding suitable functions π_i invariant under τ_i. We shall also impose the reality condition

$$\bar{\pi}_2 = \pi_1 \circ \rho . \tag{3.11}$$

In general we can try $\pi_i = f + f \circ \tau_i$, for any analytic or meromorphic function f. Taking $f(t) = t$ leads to the "Zhukovsky functions",

$$z = \frac{\alpha}{2}\left(t + \frac{\mu}{t}\right), \quad \bar{w} = \frac{\beta}{2}\left(t + \frac{1}{\mu t}\right), \tag{3.12}$$

where α, β are constants. Computing $z^2, \bar{w}^2, z\bar{w}$, and eliminating t leads to the equation

$$(3.13)\qquad \frac{4}{\alpha\beta}\left(\mu+\frac{1}{\mu}\right)z\bar{w}-4\left(\frac{1}{\mu\alpha^2}z^2+\frac{\mu}{\beta^2}\bar{w}^2\right)=\left(\mu-\frac{1}{\mu}\right)^2.$$

Next we choose the constants so that (3.11) holds. For the case (3.8) we take $\bar{\beta}=\alpha\mu$, $\alpha=1$, so that

$$(3.14)\qquad z=\frac{1}{2}\left(t+\frac{\mu}{t}\right),\quad \bar{w}=\frac{\mu}{2}\left(t+\frac{1}{\mu t}\right),$$

and (3.13) becomes (2.6) with

$$(3.15)\qquad B=\frac{4(1+\mu^2)}{(1-\mu^2)^2},\quad A=\frac{4\mu}{(1-\mu^2)^2},\quad B-2A=\frac{4}{1+\mu^2}.$$

Since the last two numbers are positive, we have an ellipse with foci on the real axis.

For the case (3.9) we choose $\beta=\bar{\alpha}$, and $\alpha=\bar{\lambda}$, so that the coefficients of z^2 and $\bar{w}^2$ in (3.16) are equal. We get

$$(3.16)\qquad z=\frac{\bar{\lambda}}{2}\left(t+\frac{\mu}{t}\right),\quad \bar{w}=\frac{\lambda}{2}\left(t+\frac{1}{\mu t}\right),$$

and equation (2.6) with

$$(3.17)\qquad B=\frac{4(\mu+\bar{\mu})}{(\mu-\bar{\mu})^2},\quad A=\frac{4}{(\mu-\bar{\mu})^2},\quad B-2A=\frac{4(\mu+\bar{\mu}-2)}{(\mu-\bar{\mu})^2}.$$

It follows that $A<0$, and $B-2A>0$, since $-2<\mu+\bar{\mu}<2$, by (3.9). Thus we have a hyperbola with foci on the real axis.

In the parabolic case we may assume that $q_1=q_2=\infty$, and $p_1=1$, $p_2=-1$. Then

$$(3.18)\qquad \tau_1(t)=-t+2,\quad \tau_2(t)=-t-2.$$

If we take

$$(3.19)\qquad \rho(t)=-\bar{t},$$

then $\tau_2=\rho\tau_1\rho$. We can satisfy (3.13) and (3.14) if we take $\pi_1=f+f\circ\tau_1$, where $f\circ\rho=\bar{f}$. Thus we take $f(t)=\alpha t^2$, $\alpha=\bar{\alpha}$,

$$(3.20)\qquad z=2\alpha(t-1)^2,\quad \bar{w}=2\alpha(t+1)^2.$$

Adding and subtracting to eliminate t gives

$$r(z, \bar{w}) \equiv (z - \bar{w})^2 - 16\alpha(z + \bar{w}) + 64\alpha^2 = 0 \,, \tag{3.21}$$

which is (2.18) with $\alpha = 4a$.

REMARK. In the above examples we chose the simplest non-trivial rational functions $f(t)$, which led us back to the examples of section 2. Other choices of f would lead to more complicated rational curves.

4. Riemann maps

The deeper geometric and analytic properties of a simply connected proper subdomain $D \subset \mathbf{C}$ are brought out in the problem of mapping it conformally onto the unit disc Δ, or right half plane H. In this section we shall indicate by example what role double valued reflection plays in this problem.

Thus, let the boundary ∂D be a branch of a real algebraic curve admitting double valued reflection. The Riemann map, $f: D \to \Delta$, continues to some neighborhood of the closure $\bar{D}$, and so maps a curve with double valued reflection to one with single valued reflection. This forces f to possess additional symmetry properties. Roughly speaking, if f could be continued globally, then the two reflected points of any point z would have to map to the single reflected point of $f(z)$. This is decisive in determining an explicit expression for f.

We first consider the domain D inside the ellipse (2.2). The first map, $z = \pi_1(t)$, in (3.14) takes the annulus $A_1^\mu = \{1 < |t| < \mu\}$ onto D, as a two fold covering

$$\pi_1 : A_1^\mu \to D \,, \tag{4.1}$$

branched at the points $t = \pm\lambda \in A_1^\mu$. We have

$$\pi^{-1}(\gamma) = \partial A_1^\mu = \gamma_1 \cup \gamma_\mu \,, \tag{4.2}$$

where γ_1 is the fixed point set of ρ, and $\gamma_\mu = \tau_1(\gamma_1)$ is the fixed point set of $\rho_\mu = \tau_1 \rho \tau_1$,

$$\rho(t) = 1/\bar{t}, \quad \rho_\mu(t) = \mu/\bar{t} \,. \tag{4.3}$$

The Riemann map,

$$f: D \to H, \; \zeta = f(z) \,, \tag{4.4}$$