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is of little cognitive value in that the purpose of the procept notion is to echo the cognitive reality of how mathematical processes are compressed mentally into manipulable mental objects. This has been the focus of attention of many researchers in mathematics education both at school and university level, including for example, Piaget (1972), Greeno (1983), Davis (1984), Dubinsky (1991), Sfard (1991), Hare & Kaput (1991). The cognitive process by which processes become conceived as manipulable objects is called *encapsulation* by Dubinsky, following Piaget.

Had the definition of *procept* been a *mathematical definition*, doubtless some mathematician would have made it before. But the procept notion implies a cognitive ambiguity — the symbol can be thought of *either* as a process, *or* as a concept. This gives a great *flexibility* in thinking — using the *process* to *do* mathematics and get answers, or using the *concept* as a compressed mental object to *think about* mathematics — in the sense of Thurston:

> I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is ${}^{134}/_{29}$ (and so forth). What a tremendous labor-saving device! To me, '134 divided by 29' meant a certain tedious chore, while ${}^{134}/_{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, a/b and a divided by b are just synonyms. To him it was just a small variation in notation.
> *(Thurston, 1990, p. 847)*

I claim that the reason why mathematicians haven't made this definition is that they *think* in such a flexible ambiguous way often without consciously realising it, but their desire for final precision is such that they write in a manner which attempts to use unambiguous definitions. This leads to the modern set-theoretic basis of mathematics in which concepts are defined as *objects*. It is a superb way to systematise mathematics but is cognitively in conflict with developmental growth in which mathematical processes *become* mathematical objects through the form of compression called encapsulation.

Sequential and Procedural Compression

A mathematician puts together a number of ideas in sequence to carry out a computation or a sequence of deductions in a proof using method (3). Hadamard performs such mental actions successively focusing on images before arguments are formulated logically:

> It could be supposed a priori that the links of the argument exist in full consciousness, the corresponding images being thought of by the subconscious. My personal introspection undoubtedly leads me to

> the contrary conclusion: my consciousness is focused on the successive images, or more exactly, on the global image; the arguments themselves wait, so to speak, in the antechamber to be introduced at the beginning of the "precising" phase. *(Hadamard, 1945, 80-81)*

Students who have little of this internal structure see in a proof just a sequence of steps which they feel forced to commit to memory for an examination:

> Maths courses, having a habit of losing every student by the end of the first lecture, definitely create a certain amount of negative feeling (as well as a considerable amount of apathy) and the aim for the exam becomes the anti-goal of 'aiming to get through so I don't have to retake' rather than the goal of 'working hard to do well because I enjoy the subject'. *(Female mathematics undergraduate, 2nd Year)*

This use of memory for routinizing sequential procedures is a valuable human tool when the mental objects to be manipulated will not all fit in the focus of attention at the same time. The memory scratch-pad available is small — about 7 ± 2 items according to Miller (1956).

When individuals fail to perform the compression satisfactorily they do not have mental objects which can be held simultaneously in memory (Linchevski & Sfard, 1991). They are then forced into using method (3) as a *defence* mechanism — remembering routine procedures and internalising them so that they need less conscious memory to process. The problem is that such procedures can only be performed in time one after another, leading to an inflexible *procedural* view of mathematics. Such procedural learning may work at one level in routine examples, but it produces an escalating degree of difficulty at successive stages because it is more difficult to co-ordinate processes than manipulate concepts. *The failing student fails because he or she is doing a different kind of mathematics which is harder than the flexible thinking of the successful mathematician.*

THE TRANSITION TO FORMAL MATHEMATICS

Students usually find formal mathematics in conflict with their experience. It is no longer about procepts — symbols representing a process to be computed or manipulated to give a result. The concepts in formal mathematics are no longer related so directly to objects in the real world. Instead the mathematics has been systematised (à la Bourbaki) and presented as a polished theory in