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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **42 (1996)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **17.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-87870

L'Enseignement Mathématique, t. 42 (1996), p. 25-48

DOUBLE VALUED REFLECTION IN THE COMPLEX PLANE

by S.M. WEBSTER [1])

INTRODUCTION

Single valued global reflection in straight lines and circles in the complex plane, introduced into function theory and developed mainly by H.A. Schwarz, has been used to great effect. Local reflection in real analytic curves as given by Schwarz and Caratheodory also plays a very important role. To many such curves one can associate global reflections, which are however multiple valued. While Caratheodory, for example, was certainly aware of this for the ellipse, the systematic theory of multiple valued reflection in one complex dimension seems to have gone undeveloped until now.

In this paper we consider the simplest curves γ in the complex plane $\mathbf{C}$ which are invariant under a double valued reflection. These include the conics, which are considered in section 2, and certain cubic and quartic curves given in section 1. We then characterize the situation intrinsically in section 3. The data consists of a pair $\{\tau_1, \tau_2\}$ of holomorphic involutions on the complexification Γ of γ, with $\tau_2 = \rho\tau_1\rho$, where ρ is the anti-holomorphic involution of Γ fixing γ. Actually, double valued reflections were first studied in several complex variables, in the context of non-degenerate complex tangents [8]. The importance of studying the dynamics of the *reversible* map $\sigma = \tau_1\tau_2$ was brought out in [8]. Some of the simpler results of [8] are used here to classify involution pairs when Γ is the Riemann sphere. As an application we show how some of the classically known Riemann maps, in somewhat different form, follow systematically from our theory.

We then go on to consider involutions on a one dimensional complex torus in sections 5 and 6. This is mostly classical in nature, although the realization by explicit algebraic equations of the kind needed here does not seem to be in the literature. In the final section we consider a special case

[1]) Partially supported by NSF grants DMS-9201966 and DMS-9504452.

involving a rectangular lattice. We give the Riemann map for the domain bounded by one branch of the associated real quartic (elliptic) curve in terms of a Weierstrass sigma quotient and the $\mathscr{P}$-function.

1. Double valued reflection

To place our work in context, we first consider, somewhat informally, the general concept of an anti-holomorphic involutive correspondence, or multiple valued reflection, on a complex manifold $\mathscr{U}$. This is an assignment $z \mapsto Q_z$, of a complex subvariety $Q_z \subset \mathscr{U}$ to each point $z \in \mathscr{U}$, such that

$$w \in Q_z \Leftrightarrow z \in Q_w \, . \tag{1.1}$$

The variety Q_z depends antiholomorphically on the point z in a way which can be made precise. The "fixed point set" is the set

$$\gamma = \{z \in \mathscr{U} \mid z \in Q_z\} \, . \tag{1.2}$$

Such a correspondence is double valued if each Q_z is generically zero dimensional and contains two points. Starting from a generic point $z_0 \in \mathscr{U}$, we have $Q_{z_0} = \{z_1, z_1'\}$. Choosing z_1, we get $Q_{z_1} = \{z_0, z_2\}$, $Q_{z_2} = \{z_1, z_3\}, \ldots$ Thus we generate a sequence

$$z_0 \mapsto z_1 \mapsto z_2 \mapsto z_3 \cdots , \tag{1.3}$$

with z_{2k} and z_{2k+1} locally determined and depending holomorphically, respectively, antiholomorphically, on z_0. If we choose z_1', then $Q_{z_1'} = \{z_0, z_2'\}$, $Q_{z_2'} = \{z_1', z_3'\}, \ldots$, and we generate a similar sequence

$$z_0 \mapsto z_1' \mapsto z_2' \mapsto z_3' \cdots . \tag{1.4}$$

A basic problem of the theory is to understand the dynamics of this process. We shall make the foregoing more precise, but only in the case where $\mathscr{U}$ is an open subset of the complex plane.

Let $r(z, \zeta)$ be holomorphic on $\mathscr{U} \times \bar{\mathscr{U}}$ where

$$\mathscr{U} \subseteq \mathbf{C}, \ \bar{\mathscr{U}} = \{\bar{z} \mid z \in \mathscr{U}\} , \tag{1.5}$$

which satisfies

$$r \circ \rho = \bar{r}, \ \rho(z, \zeta) = (\bar{\zeta}, \bar{z}) \, . \tag{1.6}$$

We set

(1.7) $$\Gamma = \{(z, \zeta) \in \mathcal{U} \times \bar{\mathcal{U}} \mid r(z, \zeta) = 0\}\,,$$

(1.8) $$\gamma = \{z \in \mathcal{U} \mid r(z, \bar{z}) = 0\}\,,$$

(1.9) $$Q_w = \{z \in \mathcal{U} \mid r(z, \bar{w}) = 0\}\,.$$

By (1.6) $r(z, \bar{z})$ is real on $\mathcal{U}$, and γ is a real analytic curve. Γ, the complexification of γ, is invariant under the anti-holomorphic involution ρ, which has $\Gamma \cap \{\zeta = \bar{z}\} \cong \gamma$ as fixed-point set. We denote the projections, restricted to Γ, by

(1.10) $$\pi_1(z, \zeta) = z, \ \pi_2(z, \zeta) = \zeta, \ \pi_2 = \overline{\pi_1 \circ \rho}\,.$$

The multiple valued reflection $z \to Q_z$ on $\mathcal{U}$ is derived from the single valued reflection ρ on Γ by

$$z \mapsto \pi_1^{-1}(z) \to \rho(\pi_1^{-1}(z)) \to \pi_1(\rho(\pi_1^{-1}(z))) \equiv Q_z\,.$$

If $z_0 \in \gamma$ and $r_z(z_0, \bar{z}_0) \neq 0$, then the holomorphic implicit function theorem gives a unique w near z_0, depending anti-holomorphically on z near z_0, and satisfying $r(z, \bar{w}) = 0$. The map $z \mapsto w$ is the local reflection in γ in the form emphasized by Caratheodory [1].We are merely considering this from a more global point of view.

DEFINITION. The real curve γ admits double valued reflection, if the two holomorphic maps $\pi_1 : \Gamma \mapsto \mathcal{U}$, $\pi_2 : \Gamma \to \bar{\mathcal{U}}$ are twofold branched coverings. (In case Γ has singularities, we may replace it by its Riemann surface in this definition.)

The maps which interchange the fibers of the maps π_i are denoted by $\tau_i : \Gamma \mapsto \Gamma$, $i = 1, 2$,

(1.11) $$\pi_i \circ \tau_i = \pi_i,\ (\tau_i)^2 = id,\ i = 1, 2;\ \tau_2 \circ \rho = \rho \circ \tau_1\,.$$

They are holomorphic maps which don't commute in general. Their commutator is σ^2,

(1.12) $$\sigma = \tau_1 \circ \tau_2\,.$$

The map σ is *reversible,* i.e. conjugate to its inverse via an involution: $\sigma^{-1} = \tau_2 \tau_1 = \tau_1 \sigma \tau_1$.

We can now explain the sequences (1.3) and (1.4) more precisely in terms of σ. From

$$r(z_0, \bar{z}_1) = r(z_2, \bar{z}_1) = r(z_2, \bar{z}_3) = 0\,,$$

we have $\tau_2(z_0, \bar{z}_1) = (z_2, \bar{z}_1)$, and

$$\sigma(z_0, \bar{z}_1) = \tau_1(z_2, \bar{z}_1) = (z_2, \bar{z}_3) .$$

Hence, the map $z_0 \mapsto z_2$ is the first component of $(z_0, \bar{z}_1) \mapsto \sigma(z_0, \bar{z}_1)$. Thus, we are led to studying the iterates σ^k of a reversible holomorphic map σ on a Riemann surface Γ.

In this paper we shall concentrate on the algebraic case. Thus, we assume $\mathscr{U} = \mathbf{C}$, and take $r(z, \zeta)$ to be a holomorphic polynomial of two complex variables. For a double valued reflection we must have

$$\deg_z r = \deg_\zeta r = 2, \deg r \leqslant 4 . \tag{1.13}$$

The condition is thus very restrictive. In fact, one can give give a complete classification. We write

$$r(z, \bar{z}) = b_0 + b_1 z\bar{z} + b_2 z^2 \bar{z}^2 + 2Re(a_0 z + a_1 z^2 + a_2 z^2 \bar{z}) , \tag{1.14}$$

where the b's are real, the a's complex, constants. The form of r is invariant under linear transformation $z \mapsto cz + d$. The form of the equation $r = 0$ is also invariant under inversion

$$z \mapsto z^{-1}, r(z, \bar{z}) \mapsto (z\bar{z})^2 r(z^{-1}, \bar{z}^{-1}) , \tag{1.15}$$

and hence, under all Moebius transformations. (1.15) results in the change of coefficients

$$(b_0, b_1, b_2; a_0, a_1, a_2) \mapsto (b_2, b_1, b_0; \bar{a}_2, \bar{a}_1, \bar{a}_0) . \tag{1.16}$$

The family of curves $r = 0$ includes the conics, the lemniscates (inversions of hyperbolas), cuspidal cubics (inversions of parabolas), as well as certain elliptic curves.

If we assume that γ is non-empty, then a translation results in $b_0 = 0$. An inversion then results in $b_2 = 0$. We assume that $\deg r = 3$, so that $a_2 \neq 0$ (otherwise, we have a conic, which case we shall treat in the next section). If both $b_0 = 0$ and $a_0 = 0$, we can still invert and reduce γ to a conic. Thus, we assume that either $b_0 \neq 0$, or $a_0 \neq 0$. The translation $z \mapsto z + c$ results in

$$a_1 \mapsto a_1 + \bar{c} a_2 ,$$

so that we can make $a_1 = 0$ by a unique choice of c. Now we can make the changes

$$z \mapsto cz, r \mapsto \lambda r, \lambda = \bar{\lambda} \neq 0, c \neq 0 ,$$

which result in

$$a_2 \mapsto \lambda c^2 \bar{c} a_2 .$$

We make $a_2 = 1$ and then restrict to $c = \bar{c}$, $\lambda c^3 = 1$. If $b_0 \neq 0$, we can make $b_0 = 1$, which gives the normal form, under the Moebius group,

$$r = 1 + bz\bar{z} + 2Re(az + z^2\bar{z}),\ b \in \mathbf{R},\ a \in \mathbf{C}\ . \tag{1.17}$$

If $b_0 = 0$, we have

$$a_0 \mapsto \lambda c a_0 = c^{-2} a_0\ .$$

We make $|a_0| = 1$, after which we must restrict to $c = \pm 1$, $\lambda = \mp 1$. The normal form is

$$r = bz\bar{z} + 2Re(az + z^2\bar{z}),\ |a| = 1,\ b \geqslant 0\ . \tag{1.18}$$

In summary we have proved the following.

PROPOSITION 1.1. *Suppose that the (non-empty) real algebraic curve $\gamma \subset \mathbf{C}$ admits double valued reflection. Then, under Moebius transformation γ is equivalent to a conic section, or to a curve $r = 0$, where r is given by either (1.17) or (1.18).*

A different normal form will appear later from the intrinsic point of view.

We note that if $\deg_z r = 1$, then we have a circle, and the above process reduces to transforming it to a straight line. These are the cases of global single valued reflection.

2. CONIC SECTIONS

In this section we shall describe the relevant geometry of real quadratic curves in the complex plane. This should give a clearer idea of the possible dynamics in (1.3) and (1.4). The description is only "local" in that it depends on making certain branch cuts, so that the double valued reflection falls into two single valued reflections. In the next section we shall give a more coherent treatment, essentially by passing to a two-sheeted Riemann surface, namely Γ, on which the double valued reflection becomes a single valued one, namely ρ.

The conic with foci $\pm a$, $a > 0$, and parameter $b > 0$ is given by

$$|z + a| + \varepsilon |z - a| = b,\ \varepsilon^2 = 1\ . \tag{2.1}$$

This is an ellipse if $\varepsilon = +1$, $b > 2a$, and one branch of a hyperbola if $\varepsilon = -1$, $b < 2a$. The other branch is gotten by replacing b with $-b$. Squaring and simplifying twice gives the equation

$$r(z, \bar{z}) \equiv Bz\bar{z} - A(z^2 + \bar{z}^2) - 1 = 0\,, \tag{2.2}$$

where

$$A = \frac{4a^2}{b^2(b^2 - 4a^2)}\,, \quad B = \frac{4(b^2 - 2a^2)}{b^2(b^2 - 4a^2)}\,, \tag{2.3}$$

$$a^2 = \frac{4A}{B^2 - 4A^2}\,, \quad b^2 = \frac{4}{B - 2A}\,. \tag{2.4}$$

(2.3) shows that (2.1) is an ellipse when $A > 0$ and a hyperbola when $A < 0$. (2.4) shows that $B - 2A$ is always positive, and that $A, B + 2A$, and $B^2 - 4A^2$ all have the same sign. Conversely, if (2.2) represents a conic with foci on the x-axis ($z = x + iy$), then we must have $B - 2A > 0$. It is an ellipse when $B + 2A > 0$, hyperbola when $B + 2A < 0$, with vertices at $\pm a_1$,

$$a_1 = \sqrt{B - 2A}\,. \tag{2.5}$$

The complexified conic Γ is given by ($\zeta = \bar{w}$)

$$r(z, \bar{w}) = Bz\bar{w} - A(z^2 + \bar{w}^2) - 1 = 0\,. \tag{2.6}$$

For each fixed z we get two values of $\bar{w}$, except when the discriminant

$$\Delta = (B^2 - 4A^2)z^2 - 4A \tag{2.7}$$

vanishes, i.e. when $z = \pm a$ is a focus. If we cut the z-plane from one focus to the other along a segment on the Riemann sphere $\mathbf{P}_1$, the Riemann surface $\pi_1 : \Gamma \to \mathbf{P}_1$ falls into two sheets and the double valued reflection splits into two single valued ones, which we denote by ρ_+, ρ_-. They are most easily visualized via the sine transform

$$z = a \sin t = a(\sin t' \cosh t'' + i \cos t' \sinh t''),\ t = t' + it''\,. \tag{2.8}$$

For the *ellipse* we make the cut from $+a$ to $+\infty$ along the positive real axis and from $-\infty$ to $-a$ along the negative real axis. The remaining open set is the biholomorphic image of the strip $|t'| < \pi/2$ under (2.8). For $c'' > 0$, the pair of segments $t'' = \pm c''$ are transformed into one confocal ellipse, denoted $E_{c''}$, while for $|c'| < \pi/2$, the vertical line $t' = c'$ is mapped to one branch $H_{c'}$ of a confocal hyperbola. For

$$c_0'' = \cosh^{-1}\left(\frac{a_1}{a}\right) = \cosh^{-1}\sqrt{\frac{B + 2A}{4A}}\,, \tag{2.9}$$

$E_{c_0''}$ is our original ellipse. It follows that in the t-coordinate the above mentioned reflections $\rho_\pm$ are given by

(2.10) $$\rho_\pm(t', t'') = (t', \pm 2c_0'' - t'') ,$$

while the first component of σ is given by

(2.11) $$\rho_- \circ \rho_+(t', t'') = (t', t'' - 4c_0'') .$$

Note that these maps preserve each confocal hyperbola branch, while permuting the confocal ellipses. By successive reflections in $t'' = c_0''$ and $t'' = -c_0''$, a suitable analytic function defined inside the original ellipse can be extended to a larger and larger domain, eventually to the entire cut z-plane.

Note that by reflecting the segment $t'' = c_0''$ in the segment $t'' = -c_0''$, and vice-versa, we see that to each point of $E_{c_0''}$ there is a unique point $z_1(z) \in E_{3c_0''}$ (lying on the same confocal hyperbola), with $r(z, \overline{z_1(z)}) = 0$. These points z_1 sweep out the first "self reflection" of the original ellipse.

For the *hyperbola* we cut the z-plane along the finite segment from $-a$ to $+a$ on the real axis. We set

(2.12) $$c_0' = \sin^{-1}\left(\frac{a_1}{a}\right) = \sin^{-1}\sqrt{\frac{4A}{B + 2A}} .$$

Then (2.8) maps the lines $t' = c_0'$, and $t' = \pi - c_0'$, onto $H_{c_0''}$, and the line $t' = \pi/2$ two-to-one onto $[a, +\infty)$. The two local single valued reflections are now given by

(2.13) $$\rho_+(t) = (2c_0' - t', t''), \rho_-(t) = (2(\pi - c_0') - t', t'') ,$$

in which we may restrict to $t'' > 0$, and

(2.14) $$\rho_- \circ \rho_+(t) = (t' + 2\pi - 4c_0', t'') .$$

By the 2π-periodicity of the sine function, these maps are single valued on the cut z-plane. They preserve each confocal ellipse and permute the branches of the confocal hyperbolas.

We can easily compute the maps τ_i for conics. If $(z', \bar{w}') = \tau_1(z, \bar{w})$, then $z' = z$, and $r(z, \bar{w}') = 0$, $r(z, \bar{w}) = 0$. Substracting these two equations gives

(2.15) $$\tau_1(z, \bar{w}) = (z, BA^{-1}z - \bar{w}) .$$

Similarly,

(2.16) $$\tau_2(z, \bar{w}) = (-z + BA^{-1}\bar{w}, \bar{w}) ,$$

and

$$\sigma(z, \bar{w}) = (-z + BA^{-1}\bar{w},\ -BA^{-1}z + (B^2A^{-2} - 1)\bar{w}) . \tag{2.17}$$

It is only the restrictions of these maps to $r(z, \bar{w}) = 0$ which has intrinsic meaning. We have proved the following.

PROPOSITION 2.1. *a) For the ellipse (2.1),* $\varepsilon = +1$, *let the* z*-plane be cut along the semi-infinite segments* $(-\infty, -a]$ *and* $[+a, +\infty)$ *of the real axis. Then the* z*-component of* σ *becomes single valued and preserves each confocal hyperbola branch while permuting the confocal ellipses. It is conjugate, via the sine transform (2.8), to the translation map (2.11) of the infinite strip. b) For the hyperbola,* $\varepsilon = -1$, *let the* z*-plane be cut along the finite segment* $[-a, +a]$ *of the real axis. The* z*-component of* σ *becomes single valued and preserves each confocal ellipse while permuting the confocal hyperbola branches. It is conjugate to the map covered by the translation map (2.14) on the upper half plane.*

The proposition demonstrates a certain vague principle first brought out in [8]: *elliptic geometry leads to hyperbolic dynamics,* while *hyperbolic geometry leads to elliptic dynamics.*

Finally, we consider a parabola with focus at $z = 0$, vertex at $z = a/2$, and directrix line $Re\, z = a > 0$,

$$a - Re\, z = |z| .$$

Simplifying as before, we get

$$r(z, \bar{w}) \equiv (z - \bar{w})^2 - 4a(z + \bar{w}) + 4a^2 = 0 , \tag{2.18}$$

with discriminant $\Delta = 8az$. Proceeding as before we find

$$\tau_1(z, \bar{w}) = (z, 2z - \bar{w} + 4a),\ \tau_2(z, \bar{w}) = (-z + 2\bar{w} + 4a, \bar{w}) , \tag{2.19}$$

and

$$\sigma(z, \bar{w}) = (-z + 2\bar{w} + 4a,\ -2z + 3\bar{w} + 12a) . \tag{2.20}$$

Again these maps must be restricted to the curve (2.18).

The squaring map $z = t^2$ plays the role analogous to the sine transform above. We may cut the z-plane from 0 to $+\infty$ along the real axis, and consider the reflections in $t' = \sqrt{a/2}$, $t' = -\sqrt{a/2}$, $t'' > 0$. The parabolic nature of the dynamics becomes clear.

3. THE INTRINSIC THEORY: GENUS ZERO

From the intrinsic point of view we start with a pair of holomorphic involutions $\tau_i : \Gamma \to \Gamma$, $i = 1, 2$, on an abstract Riemann surface Γ. The quotient spaces $\Gamma / \tau_i \equiv \Gamma_i$ have natural analytic structures [4], and τ_i is the covering involution for the branched covering $\pi_i : \Gamma \to \Gamma_i$. If

$$\tau_2 = \rho \tau_1 \rho \tag{3.1}$$

for an anti-holomorphic involution ρ on Γ, then there exists an anti-biholomorphic map $\hat{\rho} : \Gamma_1 \to \Gamma_2$ with $\hat{\rho} \circ \pi_1 = \pi_2 \circ \rho$. We are mainly concerned with the case $\Gamma_1 = \Gamma_2 \subseteq \mathbf{P}_1$, although one could study real analytic curves on an arbitrary Riemann surface Γ_1. If Γ is compact, and $\Gamma_1 = \mathbf{P}_1$, then Γ is hyperelliptic. The existence of the two functionally independent 2-fold branched coverings $\pi_i : \Gamma \to \mathbf{P}_1$ forces Γ to be either an elliptic or rational curve [4]. We shall restrict to these two cases, in this paper.

In the genus zero case, $\Gamma = \mathbf{P}_1$, which we consider in this section, the holomorphic involutions are fractional linear maps. A single one $\tau(t)$ can be normalized so that its fixed points are $t = 0, \infty$, and hence has the form $\tau(t) = -t$. The theory of a pair of such involutions is still elementary, but somewhat involved, so we shall refer to [8] for some details.

For a pair of holomorphic involutions τ_1, τ_2, let the fixed-point sets be

$$FP(\tau_i) = \{p_i, q_i\}, \quad i = 1, 2 \, . \tag{3.2}$$

If τ_1 and τ_2 have the same fixed-point sets, they are equal. They have a single common fixed point in the parabolic case. We first consider the general case in which the four points $\{p_1, q_1, p_2, q_2\}$ are all distinct. We may form their cross ratio,

$$\kappa = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)} \, . \tag{3.3}$$

Interchanging τ_1 and τ_2, or p_1 with q_1, or p_2 with q_2 results in (at most) the change $\kappa \mapsto 1/\kappa$. Thus, the conditions $\kappa > 0$, $\kappa < 0$, $Re\, \kappa = 0$, $\kappa \bar{\kappa} = 1$, for example, are intrinsic conditions on the pair τ_i. The first two occur when τ_1 and τ_2 are intertwined by an anti-holomorphic involution ρ. The significance of the second two conditions is still rather mysterious at this point.

The maps τ_1, τ_2 may be represented in homogeneous coordinates $(\xi, \eta) \in \mathbf{C}^2$ for $\mathbf{P}_1$ by a pair of linear involutions. As in section 2 of [8] they may chosen as follows,

$$\begin{aligned}\tau_1(\xi,\eta) &= (\lambda\eta, \lambda^{-1}\xi), \quad \tau_2(\xi,\eta) = (\lambda^{-1}\eta, \lambda\xi), \\ \sigma(\xi,\eta) &= (\mu\xi, \mu^{-1}\eta), \quad \mu = \lambda^2 .\end{aligned} \tag{3.4}$$

In the non-homogeneous coordinate $t = \xi/\eta$,

$$\tau_1(t) = \frac{\mu}{t}, \quad \tau_2(t) = \frac{1}{\mu t}, \quad \sigma(t) = \mu^2 t . \tag{3.5}$$

Since

$$FP(\tau_1) = \{\lambda, -\lambda\}, FP(\tau_2) = \{\lambda^{-1}, -\lambda^{-1}\}, \tag{3.6}$$

we have

$$\kappa = \left(\frac{1-\mu}{1+\mu}\right)^2 . \tag{3.7}$$

An anti-holomorphic involution ρ of $\mathbf{P}_1$ is given by reflection in some circle, which is anti-linear in homogeneous coordinates. Thus, lemma 2.2 of [8] applies directly to give the following.

LEMMA 3.1. *The normal form for the triple* τ_1, τ_2, ρ, *with* $\tau_2\rho = \rho\tau_1$, *falls into two cases. The* τ_i *are still given by (3.4) or (3.5), while*

$$\lambda = \bar{\lambda} > 1, \quad \rho(\xi,\eta) = (\bar{\eta}, \bar{\xi}), \quad \rho(t) = 1/\bar{t}, \tag{3.8}$$

or

$$\lambda\bar{\lambda} = 1, \quad 0 < \arg\lambda < \pi/2, \quad \rho(\xi,\eta) = (\bar{\xi}, \bar{\eta}), \quad \rho(t) = \bar{t} . \tag{3.9}$$

(3.11) is the elliptic case with $\kappa > 0$. (3.12) is the hyperbolic case, where $\kappa < 0$.

Next we consider the problem of realizing the data τ_i by means of an analytic curve,

$$z = \pi_1(t), \quad \bar{w} = \pi_2(t), \quad \pi_i \circ \tau_i = \pi_i . \tag{3.10}$$

This amounts to finding suitable functions π_i invariant under τ_i. We shall also impose the reality condition

$$\bar{\pi}_2 = \pi_1 \circ \rho . \tag{3.11}$$

In general we can try $\pi_i = f + f \circ \tau_i$, for any analytic or meromorphic function f. Taking $f(t) = t$ leads to the "Zhukovsky functions",

$$z = \frac{\alpha}{2}\left(t + \frac{\mu}{t}\right), \quad \bar{w} = \frac{\beta}{2}\left(t + \frac{1}{\mu t}\right), \tag{3.12}$$

where α, β are constants. Computing $z^2, \bar{w}^2, z\bar{w}$, and eliminating t leads to the equation

$$\frac{4}{\alpha\beta}\left(\mu+\frac{1}{\mu}\right) z\bar{w} - 4\left(\frac{1}{\mu\alpha^2} z^2 + \frac{\mu}{\beta^2}\bar{w}^2\right) = \left(\mu - \frac{1}{\mu}\right)^2. \tag{3.13}$$

Next we choose the constants so that (3.11) holds. For the case (3.8) we take $\bar{\beta} = \alpha\mu$, $\alpha = 1$, so that

$$z = \frac{1}{2}\left(t + \frac{\mu}{t}\right), \quad \bar{w} = \frac{\mu}{2}\left(t + \frac{1}{\mu t}\right), \tag{3.14}$$

and (3.13) becomes (2.6) with

$$B = \frac{4(1+\mu^2)}{(1-\mu^2)^2}, \quad A = \frac{4\mu}{(1-\mu^2)^2}, \quad B - 2A = \frac{4}{1+\mu^2}. \tag{3.15}$$

Since the last two numbers are positive, we have an ellipse with foci on the real axis.

For the case (3.9) we choose $\beta = \bar{\alpha}$, and $\alpha = \bar{\lambda}$, so that the coefficients of z^2 and $\bar{w}^2$ in (3.16) are equal. We get

$$z = \frac{\bar{\lambda}}{2}\left(t + \frac{\mu}{t}\right), \quad \bar{w} = \frac{\lambda}{2}\left(t + \frac{1}{\mu t}\right), \tag{3.16}$$

and equation (2.6) with

$$B = \frac{4(\mu+\bar{\mu})}{(\mu-\bar{\mu})^2}, \quad A = \frac{4}{(\mu-\bar{\mu})^2}, \quad B - 2A = \frac{4(\mu+\bar{\mu}-2)}{(\mu-\bar{\mu})^2}. \tag{3.17}$$

It follows that $A < 0$, and $B - 2A > 0$, since $-2 < \mu + \bar{\mu} < 2$, by (3.9). Thus we have a hyperbola with foci on the real axis.

In the parabolic case we may assume that $q_1 = q_2 = \infty$, and $p_1 = 1$, $p_2 = -1$. Then

$$\tau_1(t) = -t + 2, \quad \tau_2(t) = -t - 2. \tag{3.18}$$

If we take

$$\rho(t) = -\bar{t}, \tag{3.19}$$

then $\tau_2 = \rho\tau_1\rho$. We can satisfy (3.13) and (3.14) if we take $\pi_1 = f + f \circ \tau_1$, where $f \circ \rho = \bar{f}$. Thus we take $f(t) = \alpha t^2, \alpha = \bar{\alpha}$,

$$z = 2\alpha(t-1)^2, \quad \bar{w} = 2\alpha(t+1)^2. \tag{3.20}$$

Adding and subtracting to eliminate t gives

$$r(z, \bar{w}) \equiv (z - \bar{w})^2 - 16\alpha(z + \bar{w}) + 64\alpha^2 = 0\,, \tag{3.21}$$

which is (2.18) with $\alpha = 4a$.

REMARK. In the above examples we chose the simplest non-trivial rational functions $f(t)$, which led us back to the examples of section 2. Other choices of f would lead to more complicated rational curves.

4. RIEMANN MAPS

The deeper geometric and analytic properties of a simply connected proper subdomain $D \subset \mathbf{C}$ are brought out in the problem of mapping it conformally onto the unit disc Δ, or right half plane H. In this section we shall indicate by example what role double valued reflection plays in this problem.

Thus, let the boundary ∂D be a branch of a real algebraic curve admitting double valued reflection. The Riemann map, $f: D \to \Delta$, continues to some neighborhood of the closure $\bar{D}$, and so maps a curve with double valued reflection to one with single valued reflection. This forces f to possess additional symmetry properties. Roughly speaking, if f could be continued globally, then the two reflected points of any point z would have to map to the single reflected point of $f(z)$. This is decisive in determining an explicit expression for f.

We first consider the domain D inside the ellipse (2.2). The first map, $z = \pi_1(t)$, in (3.14) takes the annulus $A_1^\mu = \{1 < |t| < \mu\}$ onto D, as a two fold covering

$$\pi_1 : A_1^\mu \to D\,, \tag{4.1}$$

branched at the points $t = \pm\lambda \in A_1^\mu$. We have

$$\pi^{-1}(\gamma) = \partial A_1^\mu = \gamma_1 \cup \gamma_\mu\,, \tag{4.2}$$

where γ_1 is the fixed point set of ρ, and $\gamma_\mu = \tau_1(\gamma_1)$ is the fixed point set of $\rho_\mu = \tau_1 \rho \tau_1$,

$$\rho(t) = 1/\bar{t}, \quad \rho_\mu(t) = \mu/\bar{t}\,. \tag{4.3}$$

The Riemann map,

$$f: D \to H, \zeta = f(z)\,, \tag{4.4}$$

will, of course, be as given by H. A. Schwarz [10], [9], except that we choose to map to the right half plane $H: Re\,\zeta > 0$, rather than to the unit disc. We assume that f maps the vertex $-a_1$ to 0, and the vertex $+a_1$ to ∞. It follows that $h \equiv f \circ \pi_1$ will have simple zeros at $\pi_1^{-1}(-a_1)$ and simple poles at $\pi_1^{-1}(a_1)$. Since h is purely imaginary on the boundary of A_1^{μ}, we can extend it to successively larger and larger annuli by the two reflections

$$h = \hat{\rho} \circ h \circ \rho, \quad h = \hat{\rho} \circ h \circ \rho_{\mu}, \quad \hat{\rho}(\zeta) = -\bar{\zeta}. \tag{4.5}$$

It follows by (3.1) that the extended function h must satisfy

$$h = h \circ \rho_{\mu} \circ \rho = h \circ (\tau_1 \rho \tau_1) \circ \rho = h \circ \sigma. \tag{4.6}$$

This extended function h must also remain invariant under τ_1 by analytic continuation of functional relations. Equivalently, h is invariant under both τ_1 and τ_2. Hence, we seek $h(t)$ meromorphic for $0 < |t| < \infty$, satisfying

$$h \circ \sigma(t) = h(\mu^2 t) = h(t),$$
$$h \circ \tau_1(t) = h(\mu/t) = h(t),$$

and having simple zeros at $t = -1$, $-\mu$, and simple poles at $t = +1$, $+\mu$, $(\mu > 1)$.

We set

$$t = e^s, \quad \varphi(s) = h(e^s). \tag{4.7}$$

Then φ is to be doubly periodic with respect to the lattice

$$\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbf{Z}\}, \tag{4.8}$$

where

$$\omega_1 = 2\log\mu > 0, \quad \omega_2 = 2\pi i. \tag{4.9}$$

It is to have simple zeros at points congruent, mod Λ, to

$$a_1 = 2\pi i, \quad a_2 = \omega_1/2, \tag{4.10}$$

and simple poles at points congruent to

$$b_1 = \pi i, \quad b_2 = \pi i + \omega_1/2. \tag{4.11}$$

Since $a_1 + a_2 = b_1 + b_2$, φ can be represented as the Weierstrass Sigma quotient [5], [6]

$$\varphi(s) = c\,\frac{S(s-a_1)\,S(s-a_2)}{S(s-b_1)\,S(s-b_2)}, \tag{4.12}$$

where

$$(4.13) \qquad S(s) = s \prod_{\omega \in \Lambda - \{0\}} \left(1 - \frac{s}{\omega}\right) \exp\left(\frac{s}{\omega} + \frac{1}{2}\left(\frac{s}{\omega}\right)^2\right).$$

The map τ_1 is covered by the map

$$(4.14) \qquad \tilde{\tau}_1(s) = -s + \omega_1/2 .$$

φ and $\varphi \circ \tilde{\tau}_1$ have poles and zeros at the same points, with the same orders. Hence, $\varphi \circ \tilde{\tau}_1/\varphi = c_1$, where $c_1^2 = 1$, since τ_1 is an involution. Since the sum of the residues at the two poles of each is zero, one can see, using the form (4.14), that $c_1 = +1$. Hence, φ is automatically $\tilde{\tau}_1$-invariant. We have proved the following equivalent of the theorem of Schwarz [10].

THEOREM 4.1. *The Riemann map (4.4) of the ellipse D onto the right half plane H has the form*

$$(4.15) \qquad f(z) = \varphi(\log(z \pm \sqrt{z^2 - \mu})) ,$$

where φ is given by (4.12), (4.13).

As another example we consider the conformal map f from the domain D to the right of the right branch of the hyperbola (2.1) onto the right half plane H. That this problem is more "unstable" than the previous one may be seen by making the inversion $z \mapsto 1/z$. The hyperbola goes into the lemniscate

$$(4.16) \qquad Bz\bar{z} - A(z^2 + \bar{z}^2) = z^2\bar{z}^2 ,$$

and D goes into one of the bounded domains $\tilde{D}$ which (4.16) bounds. $\tilde{D}$ has a corner at 0 with angle ε,

$$(4.17) \qquad \tan \varepsilon = \sqrt{\frac{B - 2A}{-B - 2A}} .$$

The mapping problem is rather sensitive to the rationality properties of ε relative to π. The two branches of the lemniscate at 0 lift to different sheets of the branched covering $z = \pi_1(t)$, (3.19), the whole curve being the image of the real t-axis.

π_1 maps $t > 0$ onto the right branch of the hyperbola, i.e. onto ∂D. The sector

$$(4.18) \qquad S_0^{2\alpha} = \{0 < \arg t < 2\alpha = \arg \mu\}, \lambda = e^{i\alpha}$$

is mapped 2-to-1 onto D, with the branch point λ going to the focus $z = a$. If we commence to extend $h \equiv f \circ \pi_1$ by reflecting in the sides of $S_0^{2\alpha}$, we are likely to get a multiple valued map. Hence, we set

$$t = e^{2\alpha s/\pi},\ s = s_1 + is_2,\ \varphi(s) = h(e^{2\alpha s/\pi})\ , \tag{4.19}$$

so that $0 < s_2 < \pi$ is mapped to $S_0^{2\alpha}$. The maps σ and τ_1 are covered by

$$\tilde{\sigma}(s) = s + 2\pi i,\ \tilde{\tau}_1(s) = -s + \pi i\ . \tag{4.20}$$

Thus, φ is $2\pi i$-periodic, purely imaginary for $Im\ s = 0, \pi$, and $\varphi(-s + \pi i) = \varphi(s)$. The function $\varphi(s) = \sin(is)$ satisfies these conditions. Thus, [9]

$$f(z) = \sin\left(\frac{\pi i}{2\alpha}\log[\lambda(z \pm \sqrt{z^2 - 1})]\right)\ . \tag{4.21}$$

If $\frac{2\alpha}{\pi} = \frac{p}{q}$ is rational, then one can avoid the transcendental functions in (4.21). We set

$$t = s^{p/q},\ \varphi(s) = h(s^{p/q})\ . \tag{4.22}$$

Then φ reflects across the real axis and has simple zeros at $s = \pm 1$ and simple poles at $s = 0, \infty$. Thus, $\varphi(s) = ci(s - s^{-1})$, and we get

$$f(z) = ci[(\lambda(z \pm \sqrt{z^2 - 1}))^{q/p} - (\lambda(z \pm \sqrt{z^2 - 1}))^{p/q}]\ . \tag{4.23}$$

All the above maps are, of course, well known. The point here is that they follow naturally from our theory, as also does the Riemann map of the inside of a parabola using (3.20). One might hope to "explain" all such explicit maps within the current framework.

In place of the Riemann map we may consider the Green's function. We briefly consider the case of the ellipse D. Let $\tilde{G}(t, t_0)$ be the Green's function for A_1^μ, with pole at t_0. We have

$$\tilde{G}(\tau_1(t), \tau_1(t_0)) = \tilde{G}(t, t_0)\ , \tag{4.24}$$

since τ_1 is an *involutive* automorphism of A_1^μ. It follows that

$$\begin{aligned} G(t, t_0) &= \frac{1}{2}[\tilde{G}(t, t_0) + \tilde{G}(t, \tau_1(t_0))] \\ &= \frac{1}{2}[\tilde{G}(t, t_0) + \tilde{G}(\tau_1(t), t_0)] \end{aligned} \tag{4.25}$$

must descend to the Green's function of E^0. For the annulus $\tilde{G}$ may be constructed, for example, by the method of electrostatic images, using the reflections (4.3) in the boundary circles of A_1^μ (see [2], [7]).

The lemniscate (4.16) may serve as a useful model for domains with corners.

5. Involutions on a torus

We return to the situation at the beginning of section 3, but with a non-simply connected Riemann surface Γ. Let $\pi : \tilde{\Gamma} \to \Gamma$ be the universal covering space, and $\Lambda \subseteq Aut(\tilde{\Gamma})$ be the group of covering transformations. We consider liftings

$$\tilde{\tau}_i, \tilde{\rho} : \tilde{\Gamma} \to \tilde{\Gamma} \tag{5.1}$$

of τ_i, ρ. For each $\gamma \in \Lambda$ there is a $\gamma_1 \in \Lambda$ with

$$\tilde{\rho} \circ \gamma = \gamma_1 \circ \tilde{\rho}\,, \tag{5.2}$$

and similarly for τ_i. Also

$$\tilde{\tau}_i^2, \tilde{\rho}^2 \in \Lambda\,. \tag{5.3}$$

In this section we take $\tilde{\Gamma} = \mathbf{C}$, and Λ a group of translations, which we shall also identify with an additive subgroup of $(\mathbf{C}, +)$ of rank one or two over $\mathbf{Z}$. We shall determine what restrictions on Λ are forced if Γ is the complexification of a real curve admitting double valued reflection. We are, of course, interested in the corresponding objects on $\Gamma = \mathbf{C}/\Lambda$.

We drop the tilde notation and let $t \in \mathbf{C}$. In view of (5.3), we consider

$$\tau_i(t) = \varepsilon_i t + c_i, \varepsilon_i^2 = 1, (\varepsilon_i + 1)c_i \in \Lambda, i = 1, 2\,; \tag{5.4}$$

$$\sigma(t) = \tau_1\tau_2(t) = \varepsilon_1\varepsilon_2 t + c_1 + \varepsilon_1 c_2\,; \tag{5.5}$$

$$\rho(t) = a\bar{t} + b, a\bar{a} = 1, b + a\bar{b} \in \Lambda\,. \tag{5.6}$$

In case $\tau_2 = \rho\tau_1\rho$, we have

$$\varepsilon_1 = \varepsilon_2, c_2 = a(\varepsilon_1\bar{b} + \bar{c}_1) + b\,. \tag{5.7}$$

The constants c_i, b are only determined mod Λ. For each τ_i, either $\varepsilon_i = -1$ and $c_i \in \mathbf{C}$ can be arbitrary, or $\varepsilon_i = +1$ and $2c_i \in \Lambda$.

We set

$$a = e^{2\alpha i}, 0 \leqslant \alpha < \pi, \rho_\alpha(t) = a\bar{t}\,, \tag{5.8}$$
$$l_\alpha = \{\lambda e^{i\alpha} \,|\, \lambda \in \mathbf{R}\} = \{t \,|\, Re(ie^{-i\alpha}t) = 0\}\,.$$

ρ_α is the reflection the line l_α. If we apply the condition (3.3) we get

$$\rho_\alpha(\Lambda) = \Lambda \,. \tag{5.9}$$

Thus, Λ must be symmetric about l_α. Clearly, $b + a\bar{b} \in l_\alpha$, so (5.6) gives

$$b + a\bar{b} = \omega_0 \in \Lambda \cap l_\alpha \,, \tag{5.10}$$

and b lies on the line perpendicular to l_α and passing through $\frac{1}{2}\omega_0$. This line has the equation

$$2Re(e^{-i\alpha}(t - \omega_0/2)) = e^{-i\alpha}(t + a\bar{t} - \omega_0) = 0 \,. \tag{5.11}$$

If Λ satisfies (5.9) for some angle α, we choose $\omega_0 \in \Lambda \cap l_\alpha$, for example $\omega_0 = 0$. We then choose b satisfying (5.10), and construct ρ. If we replace b by $b + \omega_*, \omega_* \in \Lambda \cap l_\alpha$, then ω_0 gets replaced by $\omega_0 + 2\omega_*$. Hence, there are at most two inequivalent choices for ω_0 on l_α.

A point $t_0 \in \mathbf{C}$ represents a fixed-point of ρ if and only if it lies on a line of the form

$$t - a\bar{t} - b = \omega_0' \in \Lambda \,. \tag{5.12}$$

Since $t - a\bar{t}$ is orthogonal to $e^{i\alpha}$, ω_0' must lie on the line perpendicular to l_α and passing through $-\frac{1}{2}\omega_0$,

$$t + a\bar{t} + \omega_0 = 0 \,. \tag{5.13}$$

If there is an $\omega_0' \in \Lambda$ on this line, then the fixed-point set $FP(\rho)$ of ρ is non-empty, and is given by (5.12) for all such ω_0'. (5.12) is the line parallel to l_α and passing through $\frac{1}{2}(b + \omega_0')$; hence, there are at most two inequivalent choices of ω_0'.

First consider the very simple case

$$\Lambda = \{2\pi k i \mid k \in \mathbf{Z}\} \,. \tag{5.14}$$

From (5.9) we can only have $\alpha = 0$, or $\alpha = \pi/2$. In the first case, l_α is the real axis, $a = 1$, and $\omega_0 = 0$, $b = ib_2$ is purely imaginary. We may take $\omega_0' = 2ki$, $k = 1, 2$; thus

$$\rho(t) = \bar{t} + ib_2, \; FP(\rho) = \{Im\ t = b_2/2\} \cup \{Im\ t = b_2/2 + \pi\} \,. \tag{5.15}$$

In the second case l_α is the imaginary axis, $a = -1$, and we may take either $\omega_0 = 0$, or $\omega_0 = 2\pi i$. Then, either $b = b_1 \in \mathbf{R}$, or $b = b_1 + i\pi$. In the first case we have $\omega_0' = 0$, while in the second case there is no ω_0'. Thus,

$$\rho(t) = -\bar{t} + b_1, FP(\rho) = \{Re\, t = b_1/2\} \,. \tag{5.16}$$

Of course, $\Gamma \equiv \mathbf{C}^*$, and the covering projection $\pi : \mathbf{C} \to \Gamma$ is just $\zeta = \pi(t) = e^t$. The first choice of ρ gives reflection in the two rays $\arg \zeta = \frac{1}{2} b_2, \frac{1}{2} b_2 + \pi$. The second gives reflection in the circle $|\zeta| = e^{b_1}$. We must still make a choice of τ_1 as in (5.4), and find a "minimal" function F which is τ_1-invariant. Relative to ζ we have $\tau_1(\zeta) = \mu \zeta^{\varepsilon_1}$, $\mu = e^{c_1}$. For $\varepsilon_1 = -1$, we take $F = f + f \circ \tau_1 = \zeta + \mu \zeta^{-1}$. For $\varepsilon_1 = +1$, $c_1 = \pi i$, we take $F = f \cdot f \circ \tau_1 = -\zeta^2$. We have already used these in the case of conics.

Next we consider a rank two lattice (4.8), and after a coordinate change if necessary, choose a normalized basis $\omega_1 = 1$, $\omega_2 = \omega$,

$$(5.17) \qquad Im\, \omega > 0, \quad -\frac{1}{2} < Re\, \omega \leqslant \frac{1}{2}, \quad |\omega| \geqslant 1\,,$$

$$|\omega| = 1 \Rightarrow Re\, \omega \geqslant 0\,.$$

We consider those Λ which satisfy the reality condition (5.9) [3], [5]. Since $a = \rho_\alpha(1) \in \Lambda$, we have $a = n_1 + n_2 \omega$, and

$$(5.18) \qquad \begin{aligned} 1 = a\bar{a} &= n_1^2 + n_2^2 |\omega|^2 + 2 n_1 n_2 Re\, \omega \\ &\geqslant n_1^2 + n_2^2 - 2 |n_1 n_2 Re\, \omega| \\ &\geqslant |n_1|^2 + |n_2|^2 - |n_1 n_2| \geqslant |n_1 n_2|\,. \end{aligned}$$

There are two cases. If $n_1 n_2 = 0$, then either $a = \pm 1$, or $|\omega| = 1$ and $a = \pm \omega$. Otherwise, $|n_1| = |n_2| = 1$, and we have the equalities in (5.18). Equality in all three places implies $|\omega| = 1$, $Re\, \omega = \frac{1}{2}$, $n_1 n_2 \leqslant 0$, and $|n_1| = |n_2|$. Hence, $n_2 = -n_1 = \pm 1$, and $a = \pm(\omega - 1)$. $|\omega| = |\omega - 1| = 1$ implies that $\omega = (1 + \sqrt{3} i)/2$. If $a = \pm 1$, then both $\omega, \bar{\omega}$, and hence $2 Re\, \omega$ are in Λ. It follows that either $Re\, \omega = 0$, or $Re\, \omega = \frac{1}{2}$.

In summary we have the following classical result.

LEMMA 5.1. *Suppose that* $\mathbf{C}/\Lambda$ *admits the reflection (5.8). Then the possibilities for* Λ *and* a *are*

1. $Re\, \omega = 0$, $|\omega| > 1$, $a = \pm 1$;
2. $Re\, \omega = \frac{1}{2}$, $|\omega| > 1$, $a = \pm 1$;
3. $|\omega| = 1$, $0 < Re\, \omega < \frac{1}{2}$, $a = \pm \omega$;
4. $\omega = i$, $a = \pm 1, \pm i$;
5. $\omega = (1 + \sqrt{3} i)/2$, $a = \pm 1, \pm \omega, \pm(\omega - 1)$.

In particular, it follows that $J(\omega)$, the elliptic modular function [5], is real at ω. In each case one has to determine the possible reflections ρ, determine their fixed-point sets, and add a suitable τ_1.

We consider the rectangular case (1) of the lemma, for application in the next section. Let

$$\omega_1 = 1, \omega_2 = \omega = i\omega'', \omega'' > 1 \tag{5.19}$$

be a normalized basis. For $a = 1$, l_α is the real axis, $\omega_0 = 0$, or $\omega_0 = 1$, $b = ib_2$, or $b = \frac{1}{2} + ib_2$, $0 \leqslant b_2 < \omega''$. In the first case $\omega_0' = 0$, or $\omega_0' = \omega$, while there is no ω_0' in the second case. Thus, we have

$$\rho(t) = \bar{t} + ib_2,\ FP(\rho) = \{Im\, t = b_2/2\} \cup \{Im\, t = (b_2 + \omega'')/2\}\,. \tag{5.20}$$

For $a = -1$, l_α is the imaginary axis, $\omega_0 = 0$ or $\omega_0 = \omega$, $b = b_1$, or $b = b_1 + i\omega''/2$, $0 \leqslant b_1 < 1$. $\omega_0' = 0, 1$ in the first case, and there is no ω_0' in the second case. We have

$$\rho(t) = -\bar{t} + b_1,\ FP(\rho) = \{Re\, t = b_1/2\} \cup \{Re\, t = (b_1 + 1)/2\}\,. \tag{5.21}$$

If $\varepsilon_1 = -1$, then

$$FP(\tau_1) = \{c_1/2, (c_1 + \omega_1)/2, (c_1 + \omega_2)/2, (c_1 + \omega_1 + \omega_2)/2\}\,. \tag{5.22}$$

If we have $\varepsilon_1 = +1$, $2c_1 \in \Lambda$, $c_1 \notin \Lambda$, then τ_1 has no fixed points. τ_1 is then the deck transformation of an unbranched covering of another torus.

6. Embedding of tori

We turn to the problem of concretely realizing the data of the previous section in the main case. Given a complex torus $\Gamma = \mathbf{C}/\Lambda$, with a pair of holomorphic involutions induced by

$$\tau_i(t) = -t + c_i,\ i = 1, 2\,, \tag{6.1}$$

we look for a pair of two-fold branched coverings

$$\pi_i : \Gamma \to \mathbf{P}_1,\ \pi_i \circ \tau_i = \pi_i,\ i = 1, 2\,. \tag{6.2}$$

The problem is immediately solved by taking

$$z_i = \pi_i(t) \equiv \mathcal{P}(t - c_i/2),\ i = 1, 2\,, \tag{6.3}$$

where

$$\mathscr{P}(t) = \frac{1}{t^2} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(t-\omega)^2} - \frac{1}{\omega^2} \right) \tag{6.4}$$

is the Weierstrass $\mathscr{P}$-function [5], [6]. We set

$$\pi(t) = (\pi_1(t), \pi_2(t)) \; . \tag{6.5}$$

If $\pi(s_0) = \pi(t_0)$, $s_0 \neq t_0$, then $s_0 \equiv -t_0 + c_i$, mod Λ. Thus π will be one-to-one, as a map into $\mathbf{P}_1 \times \mathbf{P}_1$, if we assume

$$c_2 - c_1 \notin \Lambda \; . \tag{6.6}$$

To represent π as a map into $\mathbf{P}_2$ with homogeneous coordinates ζ, $z_1 = \zeta_1/\zeta_0$, $z_2 = \zeta_2/\zeta_0$, we again use the sigma function (4.13). We have [6]

$$\mathscr{P}(t) = -\partial_t^2 \log S(t) = -\frac{\Delta}{S(t)^2}, \; \Delta = S(t)S''(t) - S'(t)^2 \; . \tag{6.7}$$

Since $\Delta(0) = -S'(0)^2 \neq 0$, we may write π as

$$\begin{aligned} \zeta_0 &= S(t-c_1/2)^2 S(t-c_2/2)^2 \\ \zeta_1 &= \Delta(t-c_1/2)\, S(t-c_2/2)^2 \\ \zeta_2 &= \Delta(t-c_2/2)\, S(t-c_1/2)^2 \end{aligned} \tag{6.8}$$

The branch points of the map π_1 are given by (5.22), with $\omega_1 = 1$, and $\omega_2 = \omega$. By (6.6) the curve π has no finite singular points. Since $\pi_i(t)$ has a pole of order two at $t = c_i/2$, $i = 1, 2$; the plane curve has two cusps on the line at ∞ corresponding to these two parameter values. Such curves are considered in [3], for example.

To find the equation $G(z_1, z_2) = 0$ of this plane curve, we change the variable, $t \to t - c_1/2$, so that $G(\mathscr{P}(t), \mathscr{P}(t+c)) = 0$, where

$$c = (c_1 - c_2)/2 \; . \tag{6.9}$$

We set

$$\begin{aligned} x = \mathscr{P}(t+c), \; p = \mathscr{P}(t), \; p' = \mathscr{P}'(t), \\ \beta = \mathscr{P}(c), \; \beta' = \mathscr{P}'(c) \; . \end{aligned} \tag{6.10}$$

The addition theorem and differential equation satisfied by $\mathscr{P}$ [6] give

$$x + p + \beta = \frac{1}{4}\left(\frac{p' - \beta'}{p - \beta}\right)^2, \; p'^2 = 4p^3 - g_2 p - g_3 \; .$$

We rewrite these as

$$(p' - \beta')^2 = A(x, p), p'^2 = B(p) ,$$

and eliminate p'. This gives

$$F(x,p) \equiv F(x,p,\beta,\beta') \equiv (A - B - \beta'^2)^2 - 4\beta'^2 B = 0 . \tag{6.11}$$

Note that $A - B$ is quadratic in p, and $\beta'^2 = B(\beta)$. Since F is an even function of β', and $\mathscr{P}$ is an even function, changing c to $-c$ shows that we also have $F(p,x) = 0$. Since the coefficient of x^2 in F is $16(p-\beta)^2$, we must have

$$F(x,p) = G(x,p)\,(p-\beta)^2 .$$

Expanding in powers of $p - \beta$ gives

$$F(x,\beta) = 0, \partial_p F(x,\beta) = 0 ,$$

$$\begin{aligned} G(x,p) &= (1/2)\partial_p^2 F(x,\beta) + (1/6)\partial_p^3 F(x,\beta)\,(p-\beta) \\ &\quad + (1/24)\partial_p^4 F(x,\beta)\,(p-\beta)^2 . \end{aligned} \tag{6.12}$$

After some computation we get

$$\begin{aligned} G(z_1,z_2) &= (z_1-\beta)^2(z_2-\beta)^2 + \beta_1(z_1-\beta)\,(z_2-\beta) \\ &\quad + \beta_2(z_1+z_2-2\beta) + \beta_3 , \end{aligned} \tag{6.13}$$

where

$$\begin{aligned} \beta_1 &= -(12\beta^2 - g_2)/2, \beta_2 = -B(\beta), \\ \beta_3 &= (12\beta^2 - g_2)^2 - 3\beta B(\beta) . \end{aligned} \tag{6.14}$$

Next we consider the reality condition (3.11). From (5.9) and (6.4) we get

$$\overline{\mathscr{P}(t)} = a^2 \mathscr{P}(a\bar{t}) . \tag{6.15}$$

By definition $g_2 = 60G_2, g_3 = 140G_3$, where [6]

$$G_k = \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^{2k}} .$$

It follows from (5.9) that $\bar{G}_k = a^{2k}G_k$, so that

$$\bar{g}_2 = a^4 g_2, \bar{g}_3 = a^6 g_3 . \tag{6.16}$$

By (5.7) we have $c = (c_1 - a\bar{c}_1 - b + a\bar{b})/2$, so that $a\bar{c} = -c$. Hence,

$$\bar{\beta} = a^2\beta, \bar{\beta}_1 = a^4\beta_1, \bar{\beta}_2 = a^6\beta_2, \bar{\beta}_3 = a^8\beta_3 . \tag{6.17}$$

To satisfy (3.11) we *redefine*

$$\pi_i(t) = a\mathscr{P}(t - c_i/2)\ , \tag{6.18}$$

and set

$$G_0(z_1, z_2) = a^4 G(z_1/a, z_2/a)\ , \tag{6.19}$$

so that

$$\overline{G_0(z_1, z_2)} = G_0(\bar{z}_2, \bar{z}_1)\ . \tag{6.20}$$

In summary we have

PROPOSITION 6.1. *Let* $\Lambda = \mathbf{C}/\Lambda$ *have the holomorphic involutions (6.1) intertwined by the anti-holomorphic involution (5.6). Then* (Γ, ρ, τ_i) *is realized by the map (6.5), (6.18) onto the quartic curve* $G_0(z_1, z_2) = 0$ *given by (6.13), (6.14), (6.19). If the fixed-point set of* ρ *is non-empty, then this is the complexification of the real curve* $G_0(z, \bar{z}) = 0$.

7. A RECTANGULAR LATTICE

We consider the special case of Λ, ρ, τ_i as given in (5.19), (5.6), (6.1), with

$$a = +1, b = 0, \bar{c}_2 = c_1 = c_1' + ic_1'', c = ic_1''\ . \tag{7.1}$$

From (6.16), (6.15) it follows that g_2, g_3, β are real, and β' is purely imaginary. Thus, the coefficients $\beta_1, \beta_2, \beta_3$ of $G(z_1, z_2)$ are real. With $t = t' + it''$, we have

$$FP(\rho) = \{t'' = 0\} \cup \{t'' = \omega''/2\}\ , \tag{7.2}$$

$$\tau_1\{t'' = 0\} = \{t'' = c_1''\},\ \tau_1\{t'' = \omega''/2\} = \{t'' = c_1'' + \omega''/2\}\ . \tag{7.3}$$

Let us assume that $0 < c_1'' < \omega''/2$. Then the torus Λ is divided into four annuli

$$A_1 = \{0 < t'' < c_1''\},\ A_2 = \{c_1'' < t'' < \omega''/2\}\ ,$$
$$A_3 = \{\omega''/2 < t'' < c_1'' + \omega''/2\},\ A_4 = \{c_1'' + \omega''/2 < t'' < \omega''\}\ .$$

The fixed points of τ_1 are, by (5.22),

$$c_1/2, (c_1 + 1)/2 \in A_1\ , \tag{7.4}$$

$$(c_1 + i\omega'')/2, (c_1 + 1 + i\omega'')/2 \in A_3\ . \tag{7.5}$$

For the map $z = \pi_1(t)$ (6.3), we get

(7.6) $$D_j = \pi_1(A_j),\ C_j = \partial D_j,\ 1 \leqslant j \leqslant 4\ .$$

Then the z-plane is the disjoint union of $D_1, D_2 = D_4, D_3, C_1$, and C_3. π_1 maps each of A_2 and A_4 biholomorphically onto D_2, which is topologically an annulus with boundary $C_2 = C_1 - C_3$. π_1 also gives twofold branched coverings of A_i onto D_i, $i = 1, 3$, branched at (7.4), (7.5). In particular, D_1 is unbounded, containing $\pi_1(c_1/2) = \infty$ in its interior, and C_1 and C_3 are symmetric with respect to the real axis. $\pi_1(t)$ is real on the two horizontal lines through (7.4) and (7.5). It is also real on the two vertical lines $\{t' = c_1'/2\}$, $\{t' = (c_1' + 1)/2\}$, which intersect ∂A_1 in the points

(7.7) $$a_1 = \frac{1}{2} c_1',\ a_2 = \frac{1}{2} c_1' + ic_1'',$$

and

(7.8) $$b_1 + 1 = \frac{1}{2}(c_1' + 1),\ b_2 = \frac{1}{2}(c_1' + 1) + ic_1''\ .$$

$\pi_1(a_1) = \pi_1(a_2) \in C_1 \cap \mathbf{R}$ and $\pi_1(b_1 + 1) = \pi_1(b_2) \in C_1 \cap \mathbf{R}$ are the "vertices" of C_1.

The "annular" domain D_2 has the same conformal type as A_2, which is determined by $\frac{1}{2}\omega'' - c_1''$. This depends on both Λ and τ_1.

Finally we consider the Riemann map, $\zeta = f(z)$, of D_1 onto the right half plane H, which takes $\pi_1(a_1)$ to zero and $\pi_1(b_1 + 1)$ to ∞. We extend $f \circ \pi_1(t)$ to the entire t-plane by reflection in the lines $\{t'' = 0\}$ and $\{t'' = c_1''\}$. This gives a doubly periodic meromorphic function $\hat{\varphi}$ with period module

(7.9) $$\hat{\Lambda} = \{n_1 \cdot 1 + n_2 \cdot 2c_1''i \mid n_1, n_2 \in \mathbf{Z}\}\ .$$

$\hat{\varphi}$ has the representation in terms of the sigma function $\hat{S}$ relative to $\hat{\Lambda}$,

(7.10) $$\hat{\varphi} = \frac{\hat{S}(t - a_1)\hat{S}(t - a_2)}{\hat{S}(t - b_1)\hat{S}(t - b_2)}\ .$$

The invariance $\hat{\varphi} \circ \tau_1 = \hat{\varphi}$ follows as in section 4, since $\tau_1(a_1) = a_2$, $\tau_1(b_1) = b_2$. Since $z = \mathscr{P}(t - c_1/2)$, we have

THEOREM 7.1. *Let Λ be the lattice with periods (5.19), and let $D_1 \subset \mathbf{P}_1$ be the simply connected domain above. The Riemann map, $\zeta = f(z)$, of D_1 onto the right half ζ-plane H is given by*

(7.11) $$\zeta = \hat{\varphi}(\mathscr{P}^{-1}(z) + c_1/2)\ ,$$

where $\hat{\varphi}$ *is the sigma quotient (7.10) relative to the lattice (7.9), and* $\mathcal{P}^{-1}(z)$ *is the elliptic integral of the first kind, in Weierstrass normal form, relative to* Λ.

REMARK. We have seen that double valued reflection places a severe restriction on a real algebraic curve in the complex plane. In fact our results should provide the basis for a complete and explicit classification. We have also seen how double valued reflection may be used to explicitly determine Riemann maps. Apparently, all known such examples can be so explained. The result in the above theorem seems to be new. It would be interesting to work out more examples in the genus one case.

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(Reçu le 15 mai 1995)

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