2. Factorizability and factor equivalence

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Finally, we give two applications in Section 5 that show how these results are related to more concrete questions in algebraic number theory. First we indicate how to do certain index computations for rings of integers in abelian extensions of number fields. For a bicyclic quartic field this implies that the lattice generated by its quadratic integers has index 2 in the ring of integers. Then we explain that the result for units gives a method to obtain class number inequalities between so-called "arithmetically equivalent" number fields.

2. Factorizability and factor equivalence

Let G be a finite group. A character of G is said to be rational if it is the character of a representation of G defined over $\mathbf{Q}$. Denote the additive group of rational characters of G by $R(G)$. One can view $R(G)$ as the Grothendieck group of finitely generated $\mathbf{Q}[G]$-modules. It is the free abelian group generated by the set $X(G)$ of isomorphism classes of irreducible $\mathbf{Q}[G]$-modules.

The trivial character 1_H on a subgroup H of G induces the permutation character $1_H^G \in R(G)$, corresponding to the G-module $\mathbf{Q}[G/H]$. Let $\mathcal{S}$ denote the set of subgroups of G and let T be an abelian group. We will use multiplicative notation for the group operation on T.

(2.1) Definition. *A function $f\colon \mathcal{S} \to T$ is said to be factorizable if for every collection of integers $(a_H)_{H\in\mathcal{S}}$ with $\sum_{H\in\mathcal{S}} a_H 1_H^G = 0$ we have $\prod_{H\in\mathcal{S}} f(H)^{a_H} = 1$.*

(2.2) Examples. If G is the Galois group of an extension of number fields L/K then Galois theory gives a bijection between $\mathcal{S}$ and the set of intermediate fields of L/K. For any parameter associated to number fields one thus obtains a function on $\mathcal{S}$, and one may wonder if it is factorizable. The discriminant, zeta-function, and the odd part of the number of roots of unity in a number field, are all factorizable. The p-part of the class number for $p \nmid [L:K]$ is also factorizable; cf. [18]. The fact that the parameter hR/w is factorizable is known as "Brauer's class number relations" (see Section 4). See Kani and Rosen [10, 11] for factorizability results for curves and Jacobians.

A function $f\colon \mathcal{S} \to T$ induces a group homomorphism $f_*\colon \mathbf{Z}[\mathcal{S}] \to T$, where $\mathbf{Z}[\mathcal{S}]$ is the free abelian group generated by $\mathcal{S}$. By definition f is factorizable if and only if f_* vanishes on the kernel of the homomorphism $r\colon \mathbf{Z}[\mathcal{S}] \to R(G)$ given by $H \mapsto 1_H^G$. For abelian groups G the map r is surjective. For every group G the image of r has finite index by Artin's induction theorem [16, Ch. 13, Th. 30]. If G is abelian or T is divisible, then it follows that f is factorizable if and only if $f_* = gr$ for some homomorphism $g\colon R(G) \to T$. We then have

$$f(H) = \prod_{\chi \in X(G)} g(\chi)^{n_{H,\chi}}$$

where $n_{H,\chi}$ is the multiplicity of χ in 1_H^G, i.e., $1_H^G = \sum_\chi n_{H,\chi}\chi$. This is the factorization that the word factorizable refers to. One way to show that a function f is factorizable is by exhibiting such a map g. For instance, to show that the discriminant function in (2.2) is factorizable one lets $g(\chi)$ be the Artin conductor of χ (see [15, Ch. VI, § 3]).

Let us give another example from linear algebra. Suppose that K is a field of characteristic zero and that M is a finitely generated $K[G]$-module. Let φ be a $K[G]$-endomorphism of M. Then φ maps M^H to M^H for any subgroup H of G, and the characteristic polynomial $f(H) \in K[t]$ of the restriction $\varphi|_{M^H}$ is a factorizable function with values in $T = K(t)^*$. To see this, define $g(V)$ for any $\mathbf{Q}[G]$-module V as the characteristic polynomial of the K-linear endomorphism of $\mathrm{Hom}_{K[G]}(K \otimes_{\mathbf{Q}} V, M)$ induced by φ. Then $g\colon R(G) \to T$ is a homomorphism such that $g(1_H^G) = f(H)$. This result is also given by Kani and Rosen [11, Prop. 4.6]. It implies the following lemma.

(2.3) LEMMA. *The functions* $\dim_K(M^H) \in \mathbf{Z}$ *and* $\mathrm{Tr}(\varphi|_{M^H}) \in K$ *are factorizable. If* φ *is an automorphism then* $d_\varphi(H) = \det(\varphi|_{M^H}) \in K^*$ *is factorizable.* □

Now suppose that K is the quotient field of a Dedekind domain A and still assume that $\mathrm{char}\, K = 0$. By an A-lattice we mean a finitely generated A-module without A-torsion, or equivalently, a finitely generated projective A-module. An $A[G]$-lattice is an $A[G]$-module that as an A-module is an A-lattice. Denote the group of fractional A-ideals by $I(A)$. For two A-lattices $X \subset Y$ with $X \otimes K = Y \otimes K$ the quotient X/Y is an A-module of finite length. If the Jordan-Hölder factors of X/Y are $A/\mathfrak{p}_1, \ldots, A/\mathfrak{p}_m$ then the A-index $[Y : X]_A$ is defined to be the A-ideal $\mathfrak{p}_1 \cdots \mathfrak{p}_m$ (cf. [15, Ch. I, § 5]).

(2.4) DEFINITION. *We say that two* $A[G]$*-lattices* M *and* N *are factor equivalent if there is an* $A[G]$*-linear map* $i\colon M \to N$ *for which the following hold:*

(1) *the induced map* $M \otimes_A K \to N \otimes_A K$ *is an isomorphism;*

(2) *the index* $\left[N^H : (M)^H\right]_A \in I(A)$ *is a factorizable function of* H.

(2.5) PROPOSITION. *If N and M are factor equivalent then for any $A[G]$-linear embedding $j\colon M \rightarrowtail N$ the function $H \mapsto [N^H : j(M^H)]_A$ is factorizable.*

Proof. We have $j = \varphi i$, where i is an embedding as in (2.4) and φ is a $K[G]$-linear automorphism of $N \otimes_A K$. Using [15, Ch. III, § 1, Prop. 2] and the notation of (2.3) we see that

$$[N^H : j(M^H)]_A = d_\varphi(H) \cdot [N^H : i(M^H)]_A \ .$$

This is a product of two factorizable functions by (2.3) and by our choice of i. □

The fact that "factor equivalence" is an equivalence relation is an easy consequence of (2.5). If $\mathfrak{p}$ is a prime of K not dividing $\#G$ then condition (1) of (2.4) implies that the $\mathfrak{p}$-part of $[N^H : i(M)^H]_A$ is factorizable. One can prove this with [16, § 15.2] and [16, § 14.4, Lemma 21].

(2.6) REMARK. The definitions of factorizability given by Fröhlich [8; 9] and Burns [2] for abelian groups G are in agreement with our definitions. They also define the notion called $\mathbf{Q}$-factorizability in the abelian case, which is a stronger condition than factorizability. However, the function that one wants to be factorizable in the definition of factor equivalence automatically satisfies this stronger condition if it is factorizable. Thus, $\mathbf{Q}$-factor equivalence is the same as factor equivalence.

In [4, § 3] a factorizable function f with values in $I(\mathbf{Q})$ must also satisfy an additional condition : there should be a map g from the group of complex characters $R_{\mathbf{C}}(G)$ to $I(E)$, where E is some normal number field containing all character values of G, such that g is $\mathrm{Gal}(E/\mathbf{Q})$-equivariant, and such that $g(1_H^G)$ is the E-ideal generated by $f(H)$. It is not hard to see that this condition is satisfied by all functions that are factorizable in our sense.

3. RINGS OF INTEGERS

Let A be a Dedekind domain with quotient field K of characteristic zero and let L a Galois extension of K with Galois group G. The integral closure B of A in L is again a Dedekind domain. Assume that for all primes of L the residue class field extension is separable.