4. SUFFICIENCY FOR INDEPENDENT HOMOLOGY CLASSES

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then we can conclude that A is connected and we are done. If some R has more than two boundary components, then it contains two positive curves or two negative curves, and we can proceed as above to reduce the number of components of A.

It remains to consider the case where each component R_k of $\widehat{F}$ has exactly two boundary components of the form A_i^+ and A_j^-, where A_i and A_j are distinct components of A. In this case we conclude that we can arrange the components of A in a sequence $A_1, A_2, \ldots, A_n$, so that A_1 is homologous to A_2, A_2 is homologous to $A_3, \ldots, A_n$ is homologous to A_1. In this case, then, the number n of components is exactly the divisibility of α.

4. Sufficiency for independent homology classes

In this section we complete the proof of Theorem 2, dealing with the case of a set of homology classes consisting of independent elements.

LEMMA 4.1. *Let F be a closed orientable surface and let $\alpha_1, \ldots, \alpha_n \in H_1(F)$ be independent homology classes that span a summand of $H_1(F)$ on which the intersection pairing of F vanishes. Then there exists $\gamma \in H_1(F)$ such that $\gamma \cdot \alpha_n = 1$ and $\gamma \cdot \alpha_i = 0$ for $i < n$.*

Proof. This is a consequence of Poincaré Duality.

PROPOSITION 4.2. *Let F be a closed orientable surface and let $\alpha_1, \ldots, \alpha_n \in H_1(F)$ be independent homology classes that span a summand of $H_1(F)$ on which the intersection pairing of F vanishes. Then there exist pairwise disjoint simple closed curves $A_1, \ldots, A_n$ in F representing the homology classes $\alpha_1, \ldots, \alpha_n$.*

Proof. The proof will proceed by induction on n. The case $n = 1$ is given by Proposition 3.3.

Now inductively consider the case of $n > 1$ homology classes. By Proposition 3.3 we can find a simple closed curve A_n in F representing α_n. We claim that there is a simple closed curve B_n in F representing a homology class β_n such that B_n meets A_n in exactly one point and such that $[B_n] \cdot \alpha_i = 0$ for $i < n$. By Lemma 4.1 there is a homology class $\gamma_n \in H_1(F)$ such that $\alpha_i \cdot \gamma_n = \delta_{i,n}$. We begin by representing γ_n by a simple closed curve B transverse to A_n. By tubing together neighboring pairs of intersection of B with A_n of opposite sign we can transform B into a disjoint union B'

of simple closed curves meeting A_n in exactly one point. Now we can band together the components of B', using bands in the complement of A_n to create a closed curve B'' representing γ_n and meeting A_n in exactly one point. But B'' may now have self-intersections. We may then eliminate the self-intersections by sliding segments of B'' over A_n. This creates a simple closed curve B_n meeting A_n in exactly one point, and representing a homology class of the form $\beta_n = \gamma_n + k\alpha_n$, which proves the claim.

Now the union of the two curves A_n and B_n has a small neighborhood N of the form of a once punctured torus. Let F_n denote the result of removing N and replacing it with a disk D. Then $F_n - D = F - N \subset F$ and inclusion identifies $H_1(F_n)$ with the orthogonal complement of α_n and β_n in $H_1(F)$. Thus the homology classes $\alpha_1, \ldots, \alpha_{n-1}$ determine well-defined classes in $H_1(F_n)$, which we continue to call by the same names. By induction there are pairwise disjoint simple closed curves $A_1, \ldots, A_{n-1}$ in F_n representing the homology classes $\alpha_1, \ldots, \alpha_{n-1}$. Then these curves also live in F, determining the same homology classes, and are disjoint from the curve A_n. This completes the proof.

Here is a sketch of a standard but somewhat more learned proof of Proposition 4.2, suggested by M. Kervaire: Any basis for a self-annihilating summand of a skew-symmetric inner product space over **Z** can be extended to be part of a symplectic basis. Any two symplectic bases are related by an isometry of the inner product space. Half of a fixed standard symplectic basis is clearly represented by standard pairwise disjoint simple closed curves in a standard model of the surface. And any isometry is induced by a homeomorphism of the surface, so that the standard curves are taken to the desired curves. To see that any isometry is induced by a homeomorphism one can argue that the symplectic group is generated by certain elementary automorphisms and that these elementary automorphisms are induced by Dehn twist homeomorphisms around standard curves on the surface.

5. Disjoint simple closed curves on a planar surface

Subsequent proofs of realizability of non-independent homology classes will proceed by cutting the surface along curves representing a basis for homology until it becomes a punctured 2-sphere and then representing the remaining homology classes by disjoint curves on this planar surface. We