2. Curves of constant width in the euclidean plane

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(for $K = 0$ this follows immediately from combining Barbier's theorem with the isoperimetric inequality).

If γ is a curve of constant width $\mathcal{W}$ in S_K, we say that $p, \tilde{p} \in \gamma$ are *antipodal points* if the (intrinsic) distance between them is $\mathcal{W}$, which is to say that they realize the diameter of γ. We prove a result that was already known in the case of the euclidean plane (see [HS]):

THEOREM D. *If γ is a curve of constant width $\mathcal{W}$ in S_K such that every pair of antipodal points divides γ into two arcs of equal length (and, in the case of the sphere, if $\mathcal{W} < \frac{\pi}{\sqrt{K}}$) then γ is a circle.*

We must emphasize that, except for Lemma A, proofs are only given for *regular curves*, which for us means that they have no corners and the natural parametrization by arc-length is C^∞ (or just C^k for big enough k). By the expedient of using parallel curves, as explained in the next section, we can extend our results to curves consisting of regular pieces and a *finite* number of corners *(piecewise regular curves)*, but further extension does not seem possible using our methods.

The remainder of this article is organized as follows. In the next section we discuss curves of constant width in the familiar setting of the euclidean plane, and prove Lemma A and Barbier's theorem. Our proof of Barbier's theorem is similar to that in section 1.13 of [St], but we choose to present it here since the proof we give for S_K $(K \neq 0)$ is an elaboration of our proof for S_0.

In §3 we consider general oriented surfaces and construct systems of *geodesic parallel coordinates* suitable for dealing with our curves, proving a number of technical results about these coordinates, and also proving Theorem D. In the last section all pieces are put together to give the proofs of Theorem B and Corollary C.

2. Curves of constant width in the euclidean plane

We now review some background on convex curves; the basic reference here is [E]. Given a closed curve $\gamma \subseteq \mathbf{R}^2$, a straight line r is called a *supporting line* of γ if r touches γ at some point and γ is entirely contained in one of the closed half-planes bounded by r. One possible characterization of convex curves is the following: γ is convex if and only if through every point of γ there passes a supporting line of γ. If some boundary line of an enveloping strip Ω of γ touches γ at the point p then we say Ω is *supported* on p (of course Ω is supported on at least two points of γ).

Proof of Lemma A. The diameter of γ is by definition

$$\mathcal{D} = \max\{|p-q| : p, q \in \gamma\} = \max\{\mathcal{D}(p) : p \in \gamma\}.$$

Let p_0, p_1 be any two points in γ realizing its diameter ($|p_0 - p_1| = \mathcal{D}$), and let r_0, r_1 be the lines through p_0, p_1 which are orthogonal to the segment $\overline{p_0p_1}$. Then the set bounded by r_0, r_1 is an enveloping strip of Ω and has width $\mathcal{D}$.

Now assume γ has constant width $\mathcal{W}$. We have just shown that $\mathcal{W} = \mathcal{D}$. Given $p \in \gamma$, choose a supporting line r_0 through p, and let r_1 be the other supporting line parallel to r_0, touching γ at the point q. Then $|p-q| \geq \mathcal{W}$ (for the distance between r_0 and r_1 is $\mathcal{W}$) and hence $\mathcal{D}(p) \geq \mathcal{W}$. But we also have $\mathcal{D} \geq \mathcal{D}(p)$, and from these inequalities we obtain $\mathcal{D}(p) = \mathcal{W}$.

Now we prove the "if part". Given $p \in \gamma$, let $\tilde{p} \in \gamma$ be such that $|p - \tilde{p}| = \mathcal{W}$. Then p, $\tilde{p}$ realize the diameter of γ, and therefore the enveloping strip Ω orthogonal to $\overline{p\tilde{p}}$ has width $\mathcal{W}$. If γ has a well-defined tangent at p (i.e., if γ is *smooth* at p) then Ω is the only enveloping strip supported on p. Otherwise p is a corner of γ and the supporting lines at p vary between two extreme positions, the "left" and "right" tangents r^l and r^r, which are the limiting positions of the tangents to γ at p_n^l and p_n^r as $(p_n^l)_{n\geq 1}$ and $(p_n^r)_{n\geq 1}$ approach p from the left and from the right, respectively (convexity ensures that the points at which γ is smooth are dense in γ).

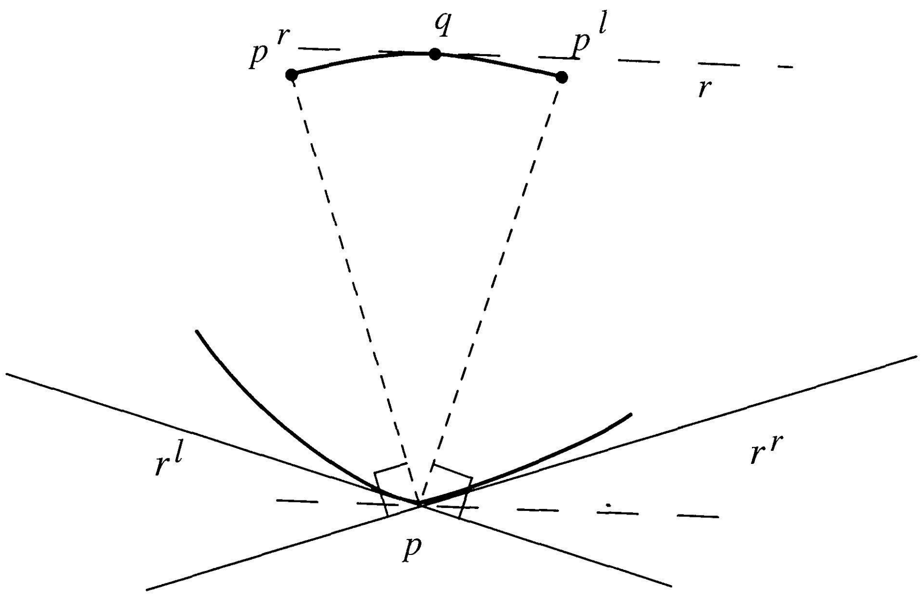

FIGURE 1

To each p_n^l there corresponds a point $\tilde{p}_n^l \in \gamma$ such that $|p_n^l - \tilde{p}_n^l| = \mathcal{W}$ and which is situated along the normal to γ at p_n^l. Therefore $p^l = \lim \tilde{p}_n^l$ is a point of γ such that $|p - p^l| = \mathcal{W}$ and which is on the line through p orthogonal to r^l; and similarly for a point p^r on the line orthogonal to r^r. It follows that the lines through p^l and p^r parallel to r^l and r^r, respectively, are supporting lines of γ. Now take an interior point q in the arc $\overset{\frown}{p^l p^r}$ of γ opposite p, and consider any supporting line r of γ at q: this line r is parallel to some supporting line through p. Hence every enveloping strip supported on q must also be supported on p, and it follows that p is the point of γ at the maximum distance $\mathcal{W}$ from q. Therefore $\overset{\frown}{p^l p^r}$ is an arc of circle with centre p and radius $\mathcal{W}$. Hence all enveloping strips supported on p have width $\mathcal{W}$.

We have thus shown that all enveloping strips of γ have width $\mathcal{W}$. □

The important point to retain from the proof of Lemma A is that, if we choose a supporting line r at a point p of a curve γ of constant width $\mathcal{W}$, then the perpendicular to r through p intersects γ at a point $\tilde{p}$ such that $|p - \tilde{p}| = \mathcal{W}$ — i.e., at an antipodal point of p.

For completeness, we observe that if γ is a simple closed curve such that $\mathcal{D}(p) = \mathcal{W}$ for all p on γ then γ is convex. For, given $p \in \gamma$, choose $\tilde{p} \in \gamma$ such that $|p - \tilde{p}| = \mathcal{W}$: γ is entirely contained in the closed disk $\mathbf{D}$ with center $\tilde{p}$ and radius $\mathcal{W}$, and the circumference $\mathcal{C}$ of $\mathbf{D}$ touches γ at the point p. Hence the tangent to $\mathcal{C}$ at p is a supporting line of γ, and we conclude that through every point of γ there passes a supporting line. This means γ is convex.

If γ is twice differentiable at p then the preceding argument shows that the absolute value $|k(p)|$ of the curvature of γ at p is *not less* than $\frac{1}{\mathcal{W}}$ (which is the curvature of $\mathcal{C}$).

For the rest of this section we consider a regular curve $\gamma(s)$ of constant width $\mathcal{W}$ and perimeter $\mathcal{L}$, parametrized by the arc length s. We define $\gamma(s)$ for all $s \in \mathbf{R}$ by letting $\gamma(s + \mathcal{L}) = \gamma(s)$, and assume that γ is traversed in the counterclockwise direction.

Let $\varphi(s)$, $s \in \mathbf{R}$, be a differentiable determination of the oriented angle between the positive x-axis and the tangent vector $\gamma'(s)$. This simply means that $\gamma'(s) = (\cos\varphi(s), \sin\varphi(s))$. The tangent vector $\gamma'(s)$ completes one counterclockwise revolution as the point $\gamma(s)$ travels once around γ, which implies that $\varphi(s + \mathcal{L}) = \varphi(s) + 2\pi$. Since γ is convex, φ is non-decreasing, so that $\varphi'(s) \geq 0$ for all $s \in \mathbf{R}$. The *signed curvature* of γ at the point $\gamma(s)$

is given by $k(s) = \varphi'(s)$, and so we have

$$\int_0^{\mathcal{L}} k(s)\,ds = \varphi(\mathcal{L}) - \varphi(0) = 2\pi. \tag{2}$$

Now define the normal vector $n(s) = \big(-\sin\varphi(s), \cos\varphi(s)\big)$, so that for each s the pair $\big(\gamma'(s), n(s)\big)$ is a positively oriented orthonormal basis of $\mathbf{R}^3$. The following formulas (special cases of Frenet's formulas) are readily verified:

$$\gamma''(s) = k(s)\,n(s), \qquad n'(s) = -k(s)\,\gamma'(s). \tag{3}$$

Let us indicate by $\Pi\big(\gamma(s)\big)$ the antipodal point of $\gamma(s)$. This map Π is an *involution* of the curve γ, in the sense that $\Pi \circ \Pi$ is the identity. By the above discussion we have

$$\Pi \circ \gamma(s) = \gamma(s) + \mathcal{W}\,n(s), \tag{4}$$

which shows that $s \mapsto \Pi \circ \gamma(s)$ is differentiable. Thus Π is an orientation-preserving diffeomorphism of the curve onto itself (if the orientation were reversed then Π would have a fixed point, which it does not). As in the case of circle diffeomorphisms, there exists a continuous function $f\colon \mathbf{R} \to \mathbf{R}$ that "lifts" Π — i.e., that satisfies the equality $\Pi \circ \gamma = \gamma \circ f$.

LEMMA E. *There exists a diffeomorphism $f\colon \mathbf{R} \to \mathbf{R}$ such that*

$$\Pi \circ \gamma(s) = \gamma \circ f(s),$$
$$f(s+\mathcal{L}) = f(s) + \mathcal{L}$$

for all $s \in \mathbf{R}$. All other liftings of Π are of the form $f + n\mathcal{L}$ for some $n \in \mathbf{Z}$.

In the case of circle diffeomorphisms, which is in essence no different from the one above, this is a standard result, and so we omit the proof of Lemma E. The uniqueness assertion has, however, some interesting consequences. For instance, from the equality $(\Pi \circ \Pi) \circ \gamma = \Pi \circ \gamma \circ f = \gamma \circ (f \circ f)$ it follows that $f \circ f(s) = s + n\mathcal{L}$, since both $f \circ f$ and the identity function are liftings of the diffeomorphism $\Pi \circ \Pi = id$.

We now prove Barbier's theorem. By Lemma E, we can rewrite (4) as $\gamma \circ f(s) = \gamma(s) + \mathcal{W}\,n(s)$. Taking the derivative of both sides and using Frenet's formulas (3) we get

$$f'(s)\,\gamma'\big(f(s)\big) = \{1 - \mathcal{W}\,k(s)\}\,\gamma'(s).$$

Since $\gamma'\big(f(s)\big)$ and $\gamma'(s)$ are unit vectors pointing in opposite directions, this implies $f'(s) = \mathcal{W}\,k(s) - 1$. Integrating and using Lemma E and (2), we finally have

$$\mathcal{L} = f(\mathcal{L}) - f(0) = \int_0^{\mathcal{L}} f'(s)\,ds$$
$$= -\mathcal{L} + \mathcal{W}\int_0^{\mathcal{L}} k(s)\,ds = -\mathcal{L} + 2\pi\mathcal{W},$$

which concludes the proof.

When we allow piecewise regular curves, the antipodal map is no longer well-defined, let alone being a diffeomorphism — and the above approach fails to work. Thus so far our proof of Barbier's theorem does not even include the simplest non-circular curve of constant width: the Reuleaux triangle.

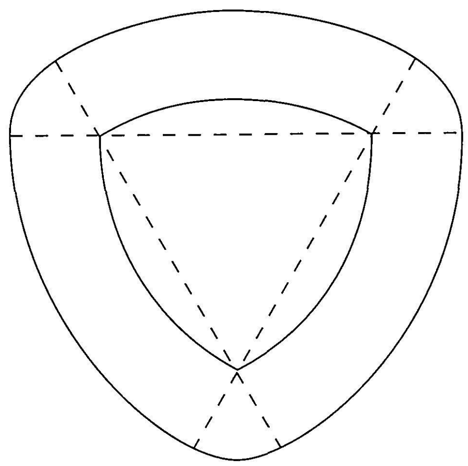

FIGURE 2
The Reuleaux triangle with one of its parallel curves

One way to overcome this deficiency is to consider *parallel curves*. If γ is a convex curve then its parallel curve γ_d surrounds γ at a fixed distance $d > 0$ from it. Assuming γ has constant width $\mathcal{W}$, γ_d has constant width $\mathcal{W} + 2d$. When γ is piecewise differentiable then each of its corners is replaced by an arc of circle with radius d in γ_d; and γ_d no longer has corners, possessing a continuously turning tangent and a continuous, piecewise differentiable angular determination $\varphi_d(s)$. With minor adaptations, our proof of Barbier's theorem shows that the perimeter of γ_d is given by

$$\mathcal{L}_d = \pi(\mathcal{W} + 2d)$$

— and letting $d \to 0$ we obtain $\mathcal{L} = \pi\mathcal{W}$ as we want.