Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 41 (1995)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HIGHER EULER CHARACTERISTICS (I)

Autor: Geoghegan, Ross / Nicas, Andrew

Kapitel: 10. Proofs of Theorems 1.1 and 1.5

DOI: https://doi.org/10.5169/seals-61816

Using (9.12),

$$\begin{aligned}
\theta'(F) &= (-1)^{qn}(p_1^4)_*(T_{M\times M}\times T_Y \cap I_*(A\times B))\\
&= (-1)^{qn}(-1)^{q(n+q)}(p_1^4)_*((T_{M\times M}\times T_Y)\cap(S_*(\Delta_M)\times \hat{F}_*([Y]\\
&\quad \times [M])\times [Y]\times [y_0]))\\
&= (-1)^q(-1)^q(p_1^4)_*((T_{M\times M}\cap(S_*(\Delta_M)\times \hat{F}_*([Y]\times[M])))\\
&\quad \times (T_Y\cap([Y]\times[y_0])))\\
&= (p_1^4)_*((T_{M\times M}\cap(S_*(\Delta_M)\times \hat{F}_*([Y]\times[M])))\times([y_0]\times[y_0]))\\
&= (p_1'')_*(T_{M\times M}\cap(S_*(\Delta_M)\times \hat{F}_*([Y]\times[M])))\\
&= \bar{I}(F)([Y]) \quad \text{by (9.11).} \quad \square
\end{aligned}$$

Combining Propositions 9.4 and 9.10 yields:

THEOREM 9.13 (Trace Formula). *The graph intersection invariant is given by:*

$$\theta'(F) = \sum_{k\geqslant 0}(-1)^k \sum_{j=1}^{N(k)} \bar{b}_j^k \cap F_*(b_j^k\times[Y]). \quad \square$$

Remark. It is easy to check that Theorem 9.13 remains valid over a principal ideal domain R in place of the coefficient field $\mathbf{F}$, provided we assume that $H_*(M; R)$ is a free R-module.

10. PROOFS OF THEOREMS 1.1 AND 1.5

In this section we prove Theorems 1.1 and 1.5 which assert the equivalence, under appropriate hypotheses, of the four definitions of the first order Euler characteristic introduced in §1.

Proof of Theorem 1.1 (ii). Let M be a compact connected oriented PL or smooth n-manifold with boundary (as well as being the underlying simplicial complex of a compatible triangulation). Using Definition A_1, we are to show that $\chi_1(M)(\gamma) = -\theta(\gamma)$; the case of other coefficient rings R will then follow immediately. Fattening if necessary, assume $n \geqslant 4$.

Let $J: M\times I \to M$ be a homotopy from id_M to a map j, such that the graph of $J|_{M\times[\frac{1}{2},1]}$ meets the graph of $p|_{M\times[\frac{1}{2},1]}$ transversely in $|\chi(M)|$ arcs; this can be achieved by classical techniques of cancelling unnecessary pairs of fixed points. Note that j will then have precisely $|\chi(M)|$ fixed points, all transverse and having the same fixed point index.

Denote by $\bar{F}^{\gamma}$, the concatenated homotopy $J^{-1} \star F^{\gamma} \star J$. It is clear that, using Definition A_1, $\operatorname{trace}(\tilde{\partial}_{k+1} D_k^{\gamma}) = \operatorname{trace}(\tilde{\partial}_{k+1} \bar{D}_k^{\gamma})$, since the new contributions cancel one another. By perturbing rel $M \times \{0, 1\}$, we may assume that the graph of $\bar{F}^{\gamma}$ meets the graph of p transversely, and that $\operatorname{Fix}(\bar{F}^{\gamma})$ consists of circles in $\overset{\circ}{M} \times (0, 1)$ and $|\chi(M)|$ arcs in $\overset{\circ}{M} \times I$ joining $\overset{\circ}{M} \times \{0\}$ to $\overset{\circ}{M} \times \{1\}$. It may be assumed (see [GN_1, §6(B)]) that $\bar{F}^{\gamma}$ is cellular with respect to suitable triangulations of M and $M \times I$.

If $\chi(M) = 0$, there are no arcs. In that case, the required geometric arguments are to be found in [GN_1, §6]; and Definitions A_1 and C_1 are indeed equivalent. (The point is that in [GN_1] there is a precise sense in which contributions to the fixed point set associated with $M \times \{0, 1\}$ are ignored, so that when such points are present, i.e. when $\chi(M) \neq 0$, something more must be said, and will now be said.)

Suppose $\chi(M) \neq 0$. $\bar{F}^{\gamma}$ is a homotopy from j to j. By our constructions, since j is homotopic to id_M and has the least possible number of transverse fixed points, all those fixed points are in the same fixed point class, (in the sense of classical Nielsen fixed point theory [Br], [J]). Moreover, the arcs are all in the same fixed point class of $\bar{F}^{\gamma}$ in the analogous sense defined in [GN_1]. By symmetry, if an arc meets $(x, 0)$ then an arc meets $(x, 1)$, but perhaps a different arc. However, since all the arcs are in the same fixed point class, the methods of [Di] allow us to perturb $\bar{F}^{\gamma}$ rel $M \times \{0, 1\}$ so that, for the perturbed map, an arc meeting $(x, 0)$ also meets $(x, 1)$. The arc $\beta(t) \equiv \bar{F}^{\gamma}(x, t)$ is homotopically trivial, for if the arc of fixed points α joins $(x, 0)$ to $(x, 1)$ then β is homotopic to $(\bar{F}^{\gamma} \circ \alpha)\,(p \circ \alpha)^{-1}$. Thus the methods of [Di] allow us to perturb $\bar{F}^{\gamma}$ further so that α is replaced by a circle of fixed points missing $M \times \{0, 1\}$ together with an arc of fixed points coinciding with β. Thus these arcs contribute zero to $\theta_R(\gamma)$. So, again, the argument in [GN_1, §6] shows, that Definitions A_1 and C_1 are equivalent: the trace formula in Definition A_1 describes the homology class of the circles.

Summarizing, we have proved Part (ii) of Theorem 1.1. $\square$

We prove Part (i) of Theorem 1.1 by first showing that Definitions B_1 and C_1 agree when X is a compact oriented manifold. Then, using the already proved Part (ii), we establish the equivalence of Definitions A_1 and B_1.

The trace formula in Definition B_1 was introduced by Knill in [Kn]. As we remarked in §1, it is independent of basis. Moreover, it is a straightforward exercise to show that it is a homotopy invariant.

Proof of Theorem 1.1(i). Let X be a finite CW complex, as in §1. By homotopy invariance of the formulas in A_1 and B_1, we may assume the attaching maps in X are polyhedral. Therefore we may PL embed X in some $\mathbf{R}^n$ as a strong deformation retract of a compact codimension 0 PL submanifold, M, e.g. a regular neighborhood. Now, any F^γ as in §1 can be extended to map $M \times S^1 \to X \hookrightarrow M$ by precomposing with $r \times \mathrm{id}$ where $r: M \to X$ is a strong deformation retraction. By Remark 9.9 and Theorem 9.13, Definitions B_1 and C_1 are equivalent for M. By Theorem 1.1(ii), Definitions A_1 and C_1 are equivalent for M. Hence, using homotopy invariance, Definitions A_1 and B_1 are equivalent for X. □

In [Kn] there also appears an "intersection class", whose definition we now recall. (Actually, the context in [Kn] is much more general: we only extract what we need.)

Throughout the remainder of this section, all homology and cohomology groups will have coefficients in the principal ideal domain R. Let M be a compatibly oriented, compact, codimension 0, PL submanifold $i: M \hookrightarrow \mathbf{R}^n$. Let $F: M \times S^1 \to M$ be such that $\mathrm{Fix}(F) \cap \partial M \times S^1 = \emptyset$. Let $[M \times S^1] \in H_{n+1}(M \times S^1, \partial M \times S^1)$ be the fundamental class of $M \times S^1$ and let $[\mathbf{R}^n]$ be the generator of $H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\})$ determined by the orientation. Following Leray [Le] and Dold [D_1], Knill defines the intersection class of F to be the image, $I_R(F)$, of $[M \times S^1]$ under the following composition:

$$H_{n+1}((M, \partial M) \times S^1) \to H_{n+1}(M \times S^1, M \times S^1 - \mathrm{Fix}(F)) \xrightarrow{(i \circ p - i \circ F, F)_*}$$
$$H_{n+1}((\mathbf{R}^n, \mathbf{R}^n - \{0\}) \times M) \xrightarrow{\cong} H_1(M)$$

where $p: M \times S^1 \to M$ is projection and $H_{n+1}((\mathbf{R}^n, \mathbf{R}^n - \{0\}) \times M) \xrightarrow{\cong} H_1(M)$ is the inverse of the isomorphism $H_1(M) \xrightarrow{\cong} H_{n+1}((\mathbf{R}^n, \mathbf{R}^n - \{0\}) \times M)$, $y \mapsto [\mathbf{R}^n] \times y$.

We make use of the following special case of [Kn, Theorem 1]:

THEOREM 10.1. *Suppose $H_*(M)$ is a free R-module. Then*

$$-I_R(F) = \sum_{k \geqslant 0} (-1)^{k+1} \sum_j \bar{b}_j^k \cap F_*(b_j^k \times [S^1])$$

where $[S^1] \in H_1(S^1)$ is the fundamental class and where for each $k \geqslant 0, \{b_j^k\}$ is a basis for $H_k(X)$ with corresponding dual basis, $\{\bar{b}_j^k\}$, for $H^k(X)$. The cap product is taken with Dold's sign convention. □

Proof of Theorem 1.5. We show $-p_* \tau(\bar{\Phi}^\gamma)_*([S^1])$ coincides with Definition B_1. As in the proof of Theorem 1.1(i) above, we may assume that X is a compact polyhedron which is PL embedded in some $\mathbf{R}^n$ as a strong deformation retract of a compact codimension 0 PL submanifold, M. Extend Φ^γ to a map $\Psi^\gamma : M \times S^1 \to X \hookrightarrow M$ by precomposing with $r \times \mathrm{id}$ where $r: M \to X$ is a strong deformation retraction. The homotopy invariance of Definition B_1 and Theorem 10.1 imply that $-I_R(\Psi^\gamma) = \chi_1(X, R)(\gamma)$. By [$D_3$, (3.3)] and [BG, §9], $I_R(\Psi^\gamma)$ coincides with $p_* \tau(\bar{\Phi}^\gamma)_*([S^1])$. □

REFERENCES

[Ba] BASS, H. Euler characteristics and characters of discrete groups. *Invent. Math. 35* (1976), 155-196.

[BG] BECKER, J.C. and D.H. GOTTLIEB. Transfer maps for fibrations and duality. *Compositio Math. 33* (1976), 107-133.

[Bi] BIERI, R. *Homological dimension of discrete groups.* Second edition, Queen Mary College, Department of Pure Mathematics, London, 1981.

[B] BROWN, K.S. *Cohomology of Groups.* Springer-Verlag, New York, 1982.

[Br] BROWN, R.F. *The Lefschetz fixed point theorem.* Scott Foresman, Chicago, 1971.

[C] COOKE, G. Replacing homotopy actions by topological actions. *Trans. Amer. Math. Soc. 237* (1978), 391-406.

[Di] DIMOVSKI, D. One-parameter fixed point indices. *Pacific J. Math. 164* (1994), 263-297.

[DG] DIMOVSKI, D. and R. GEOGHEGAN. One-parameter fixed point theory. *Forum Math. 2* (1990), 125-154.

[D_1] DOLD, A. Fixed point index and fixed point theorem for Euclidean neighborhood retracts. *Topology 4* (1965), 1-8.

[D_2] —— *Lectures on algebraic topology.* Second edition, Springer-Verlag, New York, 1980.

[D_3] —— The fixed point transfer of fibre-preserving maps. *Math. Z. 148* (1976), 215-244.

[DV] DYER, E. and A.T. VASQUEZ. An invariant for finitely generated projectives over $\mathbf{Z}G$. *J. Pure Appl. Algebra 7* (1976), 241-248.

[Eck] ECKMANN, B. Cyclic homology of groups and the Bass conjecture. *Comment. Math. Helv. 61* (1986), 193-202.

[G] GEOGHEGAN, R. The homomorphism on fundamental group induced by a homotopy idempotent having essential fixed points. *Pacific J. Math. 95* (1981), 85-93.

[GN_1] GEOGHEGAN, R. and A. NICAS. Parametrized Lefschetz-Nielsen fixed point theory and Hochschild homology traces. *Amer. J. Math. 116* (1994), 397-446.

[GN_2] GEOGHEGAN, R. and A. NICAS. Trace and torsion in the theory of flows. *Topology 33* (1994), 683-719.