5.3 3-folds with $b_2 \geq 3$

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **41 (1995)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

REMARK 14. The Hessian of a binary form $F \in S^3 H^\vee$ is identically zero iff F is degenerate; it is negative semi-definite if F is non-degenerate and $\Delta(F) \leqslant 0$; it is indefinite iff $\Delta(F) > 0$ [*Ca*]. Only in the indefinite case $\Delta(F) > 0$ can the closure $\bar{\mathscr{H}}_F := \{h \in H_\mathbf{R} \mid \det F^t(h) \leqslant 0\}$ of the Hesse cone be a proper subset of $H_\mathbf{R}$.

EXAMPLE 16. Let $P = \mathbf{P}_{\mathbf{P}^2}(E)$ be the projectivization of a rank-2 vector bundle E with Chern classes $c_i = c_i(E)$. The cup-form of P yields the cubic polynomial $f = (c_1^2 - c_2)X^2 + 3(-c_1)X^2Y + 3XY^2$ whose Hessian is $H_f = (-c_2)X^2 + c_1XY - Y^2$. Rewriting H_f as $H_f = -\frac{1}{4}[(2Y - c_1X)^2 + X^2(4c_2 - c_1^2)] = \frac{-1}{4}[(2Y - c_1X)^2 - \Delta(f)X^2]$ we find 3 possibilities for the Hesse cone:

i) $\Delta(f) < 0$: $\mathscr{H}_f = H^2(P, \mathbf{R}) \backslash \{0\}$

ii) $\Delta(f) = 0$: $\mathscr{H}_f = H^2(P, \mathbf{R}) \backslash L_{c_1}$ for a real line L_{c_1} depending on c_1 ($L_{c_1} = \mathbf{R}(2, c_1)$ in the coordinates X, Y)

iii) $\Delta(f) > 0$: $\mathscr{H}_f$ is an open cone whose angle is determined by $\Delta(f)$ $((Z + \sqrt{\Delta(f)}X)(Z - \sqrt{\Delta(f)}X) > 0$ in coordinates $X, Z := 2Y - c_1X)$.

5.3 3-FOLDS WITH $b_2 \geqslant 3$

Let X be a 1-connected, compact complex 3-fold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}^{\oplus 3}$. The cup-form of X gives rise to a curve C_X of degree 3 in the projective plane $\mathbf{P}(H^2(X, \mathbf{C}))$:

$$C_X := \{\langle h \rangle \in \mathbf{P}(H^2(X, \mathbf{C})) \mid h^3 = 0\}.$$

A first natural question is which types of plane cubic curves occur in this way?

Recall that there are 10 types of plane cubics, namely: 1) non-singular cubics, 2) irreducible cubics with a node, 3) irreducible cubics with a cusp, 4) reducible cubics consisting of a smooth conic and a transversal line, 5) smooth conics with a tangent line, 6) three lines forming a triangle, 7) three distinct lines through a common point, 8) a double line with a third skew line, 9) a triple line, 10) the trivial 'cubic' with equation 0.

LEMMA 4. *If the 3-fold X has a non-trivial Hodge number $h^{2,0}(X) \neq 0$, then C_X is of type 4), 6) 9) or 10).*

Proof. Choose basis vectors $e^{k,l} \in H^{k,l}(X)$, so that every $h \in H^2(X, \mathbf{C})$ can be uniquely written as $h = xe^{2,0} + ye^{1,1} + ze^{0,2}$.

Then clearly $h^3 = y[y^2(e^{1,1})^3 + 6xz(e^{2,0} \cdot e^{1,1} \cdot e^{0,2})]$.

We now realize the cubics of types 7)-10). These cubics are degenerate, i.e. they are cones, and therefore their Hessians vanish identically. From section 4.3 we know that they can not be realized by Kählerian 3-folds.

PROPOSITION 20. *The plane cubics of types 7)-10) can all be realized by 1-connected, non-Kählerian 3-folds.*

Proof. 'Cubics' of type 10) can be realized by elliptic fibre bundles over surfaces Y with $b_2(Y) = 5$. In order to realize cubics of type 9) or 7) one blows up one or two points in an elliptic fibre bundle over a surface with $b_2 = 4$ or 3 respectively. The realization of a type 8) cubic is a little trickier: One starts with an elliptic fibre bundle over a surface Y with $b_2(Y) = 3$, and blows up one of its fibers. The resulting 3-fold X' has $b_2(X') = 2$ and $F_{X'} \equiv 0$. Now choose a line l in the exceptional divisor E of X', and let X be the blow-up of X' along l. The cup-form of X yields the cubic polynomial $x^2[y(-3l \cdot E) - x(\deg N_{C/X'})]$ with a non-zero coefficient $-3l \cdot E = 3$.

There are four types of complex cubics which we have been able to realize by projective 3-folds.

PROPOSITION 21. *Cubics of type 1), 3), 4) and 6) are realizable by 1-connected projective 3-folds.*

Proof. Type 1) occurs for blow-ups of complete intersections in two distinct points. The product $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ realizes a triangle, whereas most projective bundles over a surface with $b_2 = 2$ lead to the union of a smooth conic and a transversal line.

Irreducible cubics with a cusp can be obtained by blowing-up a line and a point in $\mathbf{P}^3$. The resulting 3-fold yields the cubic polynomial $X^3 - 3XY^2 - 2Y^3 + Z^3 = (X + Y)^2(X - 2Y) + Z^3$.

The remaining two types of cubics are cubics with a node (type 2)), and smooth conics with a tangent line (type 5)). We do not know if these types are realizable by projective 3-folds. A non-Kählerian 3-fold whose cup-form yields a nodal cubic can be constructed: one just takes the blow-up of two suitable curves in Oguiso's Calabi-Yau 3-fold with $b_2 = 1$ and vanishing cup-form.

Finally we like to show that the non-emptiness condition on the index cone of a projective 3-fold with $h^{0,2} = 0$ gives non-trivial restrictions for the possible cup-forms if $b_2 \geqslant 4$. Further investigations of this condition will appear elsewhere [Sch].

EXAMPLE 17. Let H be a free $\mathbf{Z}$-module of rank 4 with basis $(e_i)_{i=1,\ldots,4}$. Consider a trilinear form $F \in S^3 H^\vee$ and its adjoint map $F^t : H \to S^2 H^\vee$. The image $F^t(h)$ of an element $h \in H$ is in terms of the chosen basis $(e_i)_{i=1,\ldots,4}$ represented by the symmetric 4×4-matrix $[[he_i e_j]]_{i,j=1,\ldots,4}$. Suppose this matrix is a diagonal sum $[[he_i e_j]]_{i,j=1,2} \oplus [[he_k e_l]]_{k,l=3,4}$ such that the determinants of both 2×2-matrices are negative for every $h \in H \backslash \{0\}$.

In this case $F^t(h)$ were of signature $(1, -1, 1, -1)$ for every $h \in H \backslash \{0\}$, and we would have $I_F = \mathscr{H}_F = \emptyset$.

All these conditions can be met, e.g. by setting $e_1^2 e_2 = e_2^3 = e_3^2 e_4 = e_4^3 = 1$, $e_1 e_2^2 = e_3 e_4^2 = 2$, and $e_i e_j e_k = 0$ otherwise. In this particular case the image of $h = \sum_{i=1}^4 h_i e_i$ under F^t is represented by the matrix

$$\left[\begin{array}{cc|cc} h_2 & h_1 + 2h_2 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ h_1 + 2h_2 & 2h_1 + h_2 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & h_4 & h_3 + 2h_4 \\ & & h_3 + 2h_4 & 2h_3 + h_4 \end{array}\right],$$

which has a positive determinant unless $h = 0$.

REFERENCES

[A] ARNDT, F. Zur Theorie der binären kubischen Formen. *Crelle's Journal 53,* 309-321 (1857).

[Ar1] ARONHOLD, S. Zur Theorie der homogenen Funktionen dritten Grades von drei Variabeln. *Crelle's Journal 39,* 140-159 (1850).

[Ar2] —— Theorie der homogenen Funktionen dritten Grades von drei Veränderlichen. *Crelle's Journal 55,* 93-191 (1858).

[At] ATIYAH, M. Some examples of complex manifolds. *Bonner Math. Schriften 5,* 1-28 (1958).

[A/H/S] ATIYAH, M., N. HITCHIN and I. SINGER. Self-duality in four-dimensional Riemannian geometry. *Proc. R. Soc. Lond. A. 362,* 425-461 (1978).