# 1. INTEGRAL AND ABSOLUTE CONTINUITY AND THE LOCAL PROBLEM

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## 1. Integral and absolute continuity and the local problem

Definition 1.1. *A bounded subset* $\Omega$ *of* $\mathbf{R}^n$ *is called a Lipschitz domain if for any* $a \in \Omega \setminus \mathring{\Omega}$, *there exists an open neighborhood* $U$ *of* $a$ *in* $\mathbf{R}^n$, *a coordinate system (isometric to the canonical one)* $(x', x_n) = ((x_1, \dots, x_{n-1}), x_n)$, *and a Lipschitz continuous function* $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$ *such that*

$$\Omega \cap U = \{(x', x_n); \varphi(x') \leqslant x_n\} \cap U .$$

*Also, if the new coordinates are actually obtained by permuting the canonical ones, then* $\Omega$ *is called a simple Lipschitz domain.*

Note that, the *border* of the domain $\Omega$, $b\Omega := \Omega \setminus \mathring{\Omega}$, is either the empty set or a $(n-1)$-dimensional Lipschitz submanifold of $\mathbf{R}^n$ (assumed with the standard induced orientation).

Let now $\Omega$ be a Lipschitz domain in $\mathbf{R}^n$ and $\omega$ an open set in $\mathbf{R}^{n-1}$. A locally bi-Lipschitz mapping $\varphi : \omega \to \Omega$ is called *Lipschitz embedding* provided $\varphi$ maps $\omega$ homeomorphically onto $\varphi(\omega)$. Furthermore, if $S$ is a topological space, $h : S \times \omega \to \Omega$ is called a *continuous family of Lipschitz embeddings* if $h_s := h(s, \cdot)$ is a Lipschitz embedding for each fixed $s \in S$, and if the mappings

$$S \ni s \mapsto \frac{\partial h_s}{\partial x_i} \in L^\infty(\omega, \mathrm{loc}), \quad i = 1, \dots, n-1 , \tag{1.1}$$

are continuous. Here $L^\infty(\omega,\mathrm{loc})$ is endowed with the usual (Fréchet) topology given by uniform convergence on compact subsets of $\omega$. Throughout this paper $S$ will actually always be a locally closed subspace of some $\mathbf{R}^k$.

Let $L^1(\Omega, \mathrm{loc})$ stand for the vector space of differential forms with locally integrable coefficients on $\Omega$. We consider this space endowed with the usual (locally convex) topology.

Definition 1.2. *A complex-valued* $(n-1)$*-form* $u$ *defined on* $\Omega$ *is called integrally continuous if:*

*(1) the form* $u$ *is locally* $(n-1)$*-integrable, i.e.* $\varphi^* u$ *is locally integrable on* $\omega$ *for any Lipschitz embedding* $\varphi : \omega \to \Omega$;

*(2) the mapping* $S \ni s \mapsto h_s^* u \in L^1(\omega, \mathrm{loc})$ *is continuous, for any continuous family of Lipschitz embeddings* $h : S \times \omega \to \Omega$.

Examples. Let $u = \sum_{i=1}^n (-1)^{i-1} u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ be a $(n-1)$-form on $\Omega$ where, as usual, the symbol under the "hat" is omitted in the product.

(1) If for $1 \leqslant i \leqslant n$ the functions $u_i|_C$ are $\mu_{n-1}$-integrable for any compact set $C \subset \Omega$ having $\mu_{n-1}(C) < +\infty$, then $u$ is locally $(n-1)$-integrable. In particular, this is the case if $u_i$ are locally bounded.

(2) Suppose that there exists a set $A \subset \Omega$ of zero $(n-1)$-dimensional Hausdorff measure such that $u_i|_{(\Omega \setminus A)}$ is continuous (in the induced topology) and $u_i|_{C \cap (\Omega \setminus A)}$ is $\mu_{n-1}$ integrable for any $1 \leqslant i \leqslant n$ and any compact set $C \subset \Omega$ having $\mu_{n-1}(C) < +\infty$. Then $u$ is integrally continuous as well.

(3) For $n = 1$, a $(n-1)$-form $u$ is a function and, in this case, the form $u$ is integrally continuous if and only if the function $u$ is continuous.

Recall the usual exterior derivative operator $d$. The main result of this section is the following.

THEOREM 1.3. *Consider a Lipschitz domain* $\Omega$ *in* $\mathbf{R}^n$. *Let* $u$ *be an integrally continuous* $(n-1)$-*form on* $\Omega$ *and let* $f$ *be a locally integrable n-form on* $\Omega$. *The following are equivalent.*

*(1) For any compact Lipschitz domain* $K \subseteq \Omega$ *we have* $\int_{\partial K} u = \iint_K f$.

*(2) For any rectangle* $Q \in \mathscr{R}(\Omega)$ *we have* $\int_{\partial Q} u = \iint_Q f$.

*(3)* $du = f$ *in the distribution sense on* $\mathring{\Omega}$.

Before we proceed with the proof of this theorem, we shall prove a lemma. To state it, we need some more notation. Let $\chi$ be a positive, smooth, function supported in the closed unit ball in $\mathbf{R}^n$ and normalized such that $\iint_{\mathbf{R}^n} \chi\, dx = 1$. For $\varepsilon > 0$, set $\Omega_\varepsilon := \{x \in \mathbf{R}^n; \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ and, for any $\Phi \in L^1(\mathring{\Omega}, \mathrm{loc})$, set

$$\Phi_\varepsilon(x) := \iint_{\mathbf{R}^n} \Phi(x - \varepsilon y)\, \chi(y)\, dy, \quad x \in \Omega_\varepsilon .$$

It is a well-known fact that $\Phi_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and that $\Phi_\varepsilon \to \Phi$ in $L^1(\mathring{\Omega}, \mathrm{loc})$ as $\varepsilon$ tends to zero. For a locally integrable form $u$ on $\Omega$, $u_\varepsilon$ is defined componentwise.

LEMMA 1.4. *Let* $\Omega$ *be a Lipschitz domain in* $\mathbf{R}^n$ *and let* $u$ *be an integrally continuous* $(n-1)$-*form on* $\Omega$. *Then:*

*(1)* $u \in L^1(\Omega, \mathrm{loc})$;

*(2)* $\varphi^* u_\varepsilon \to \varphi^* u$ *in* $L^1(\omega, \mathrm{loc})$ *as* $\varepsilon$ *approaches zero, for any Lipschitz embedding* $\varphi : \omega \to \mathring{\Omega}$.

*Proof.* For each sufficiently small $\varepsilon > 0$, fixed for the moment, consider the continuous family of Lipschitz embeddings

$$h: \Omega_\varepsilon \times \omega_\varepsilon \ni (x, t) \mapsto (x_1 + t_1, \ldots, x_{n-1} + t_{n-1}, x_n),$$

where $\Omega_\varepsilon$, defined above, stands for the space of parameters and $\omega_\varepsilon$ stands for a suitably small, open neighborhood of the cube $[-\varepsilon, \varepsilon]^{n-1}$. Obviously, $(h_x^* u)(t) = \Phi(x_1 + t_1, \ldots, x_{n-1} + t_{n-1}, x_n) dt_1 \wedge \cdots \wedge dt_{n-1}$, for some function $\Phi$. Since $u$ is integrally continuous, the function

$$\Phi^\varepsilon(x) := \varepsilon^{1-n} \int_{[-\varepsilon, \varepsilon]^{n-1}} \Phi(x_1 + t_1, \ldots, x_{n-1} + t_{n-1}, x_n) dt_1 \wedge \cdots \wedge dt_{n-1}$$

is continuous on $\Omega_\varepsilon$. For any small, fixed $x_n$, the Lebesgue differentiation theorem yields that $\Phi^\varepsilon(\cdot, x_n) \to \Phi(\cdot, x_n)$, as $\varepsilon \to 0$, almost everywhere with respect to the $(n-1)$-dimensional Lebesgue measure on $\{x' \in \mathbf{R}^{n-1}; (x', x_n) \in \mathring{\Omega}\}$. Using Fubini's theorem we infer that $\Phi^\varepsilon \to \Phi$, as $\varepsilon \to 0$, almost everywhere on $\Omega$. Thus, $\Phi$ is $\lambda_n$-measurable.

Next, let $Q = Q' \times Q_n$ be a rectangle in $\mathbf{R}^{n-1} \times \mathbf{R}$ which is contained in $\mathring{\Omega}$, and consider the continuous family of Lipschitz embeddings

$$k: Q_n \times Q' \ni (x_n, x') \mapsto (x', x_n) \in \Omega .$$

Hence, $k_{x_n}^* u = \Phi(\cdot, x_n) dx_1 \wedge \cdots \wedge dx_{n-1}$. As $u$ is integrally continuous, the mapping

$$Q_n \ni x_n \mapsto \int_{Q'} |\Phi(x', x_n)| \, dx_1 \wedge \cdots \wedge dx_{n-1}$$

is continuous. In particular, the iterated integral

$$\int_{Q_n} \int_{Q'} |\Phi(x', x_n)| \, dx' \wedge dx_n$$

is finite. By Fubini's theorem, it follows that $\Phi$ is integrable on $Q$.

Now, if $u = \sum_{i=1}^n (-1)^{i-1} u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ the above reasoning gives that $u_1 = \Phi$ is integrable on $Q$. Likewise, $u_2, \ldots, u_n$ are integrable on $Q$, and $u$ is thus locally integrable on $\mathring{\Omega}$.

To conclude the proof of (1) it suffices to show that any $a \in \Omega \setminus \mathring{\Omega}$ has a compact neighborhood $K$ in $\mathbf{R}^n$ such that $u$ is integrable on $K \cap \Omega$. To see this, there is no loss of generality assuming that $K$ is so that

$$K \cap \Omega = \{(x', x_n); x' \in Q', \varphi(x') \leqslant x_n \leqslant \varphi(x') + \varepsilon\},$$

where $Q'$ is a rectangle in $\mathbf{R}^{n-1}$, $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$ is a Lipschitz function, and $\varepsilon > 0$ is some fixed, sufficiently small number. This time we take the continuous family of Lipschitz embeddings

$$h' : [0, \varepsilon] \times Q' \ni (s, x') \mapsto (x', \varphi(x') + s) \in \Omega$$

and proceed as before. Hence, (1) follows.

To see (2), let $u = \sum_{i=1}^{n} (-1)^{i-1} u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ in $\Omega$, so that we have $\varphi^* u = (\sum_i (u_i \circ \varphi) \Phi_i) dt_1 \wedge \cdots \wedge dt_{n-1}$, where $\Phi_i$ are measurable functions, locally (essentially) bounded on $\omega$. Similarly, $\varphi^* u_\varepsilon = (\sum_i ((u_i)_\varepsilon \circ \varphi) \Phi_i) dt_1 \wedge \cdots \wedge dt_{n-1}$.

Given a compact subset $C$ of $\omega$, we consider the continuous family of Lipschitz embeddings $(s, t) \mapsto \varphi(t) - s$, where $t$ lies in an open neighborhood of $C$ and $s$ lies in a small open ball centered at the origin of $\mathbf{R}^n$. Let also $\theta > 0$ be an arbitrary, fixed number. By the integral continuity of $u$, there exists $\delta > 0$ such that

$$\int_C \left| \sum_{i=1}^{n} u_i(\varphi(t) - \varepsilon y) \Phi_i(t) - \sum_{i=1}^{n} u_i(\varphi(t)) \Phi_i(t) \right| \chi(y) dt < \theta \tag{1.2}$$

for all $|y| \leqslant 1$ and $0 < \varepsilon < \delta$. Since, by (1), the functions $u_i$ are locally integrable on $\mathring{\Omega}$, the function

$$\varepsilon^{-n} \sum_i u_i(y) \Phi_i(t) \chi\left(\frac{\varphi(t) - y}{\varepsilon}\right), \quad \varepsilon > 0 ,$$

is locally integrable on $\omega \times \mathring{\Omega}$. Integrating (1.2) against $dy$ over the closed unit ball in $\mathbf{R}^n$ and then changing the order of integration, we obtain

$$\int_C |\varphi^* u_\varepsilon - \varphi^* u| \, d\mu_{n-1} \leqslant c_n \theta ,$$

for some $c_n > 0$ depending only on $n$. Since $\theta > 0$ was arbitrary, the proof of the lemma is therefore complete. □

*Proof of Theorem 1.3.* Obviously (1) implies (2). Next, assume that (2) holds and let $Q$ be an arbitrary rectangle in $\mathbf{R}^n$ contained in $\mathring{\Omega}$. It is then straightforward to see that, for a sufficiently small $\varepsilon > 0$,

$$\int_{\partial Q} u_\varepsilon = \iint_Q f_\varepsilon .$$

Since $u_\varepsilon$ is smooth, the standard form of Stokes formula gives that $\iint_Q du_\varepsilon = \iint_Q f_\varepsilon$. As $Q$ was arbitrarily chosen, we see that $du_\varepsilon = f_\varepsilon$ on $\mathring{\Omega}_\varepsilon$ and, hence, by letting $\varepsilon$ go to zero, $du = f$ in the distribution sense on $\mathring{\Omega}$. Thus, (2) $\Rightarrow$ (3).

Finally, we consider the implication (3) $\Rightarrow$ (1). Using a smooth partition of unity, it is not difficult to see that matters can be reduced to verifying (1.1) in the following cases:

(i) the support of $u$ is included in the interior of $K$;

(ii) the domain $K$ has the form

$$\{(x', x_n) \in [0, 1]^{n-1} \times [0, 1]; x_n \leqslant \varphi(x')\}, \tag{1.3}$$

for some Lipschitz function $\varphi: \mathbf{R}^{n-1} \to (0, 1)$.

We present the proof in the second case, as the proof the first case goes along the same lines and is somewhat simpler. Let us first note that, if $K_\varepsilon$ is as in (1.3) except that $\varphi$ has been replaced by $\varepsilon\varphi$, with $0 < \varepsilon < 1$, on account of the integral continuity of $u$ we have

$$\int_{\partial K} u = \lim_{\varepsilon \to 1} \int_{\partial K_\varepsilon} u .$$

Hence, it suffices to prove the statement with $K_\varepsilon$ in place of $K$ or, in other words, assuming that the compact domain $K$ from (1.2) is actually contained in $\mathring{\Omega}$. Furthermore, since by (1) $du_\varepsilon = f_\varepsilon$ on $\Omega_\varepsilon$ for all $\varepsilon > 0$, and since

$$\int_{\partial K} u_\varepsilon \to \int_{\partial K} u, \quad \iint_K f_\varepsilon \to \iint_K f$$

as $\varepsilon \to 0$ (the first convergence utilizes the integral continuity of $u$), there is no loss in generality if we assume that $u$ and $f$ are smooth forms in a neighborhood of $K$.

Consider now the bi-Lipschitz homeomorphism

$$h: [0, 1]^n \ni (x', x_n) \mapsto (x', x_n \varphi(x')) \in K .$$

From the change of variable formula ([Fe3], Theorem 3.2.3, p. 243) we have

$$\int_{\partial K} u = \int_{\partial [0,1]^n} h^* u .$$

Also, a routine calculation shows that

$$(h^* u)(x', x_n) = v(x', x_n) + \left( \sum_{i=1}^{n-1} w_i(x', x_n) \partial_i \varphi(x') \right) dx_1 \wedge \cdots \wedge dx_{n-1} ,$$

where the coefficients of the $(n-1)$-form $v$ as well as $(w_i)_i$ are Lipschitz functions. Clearly, the usual Stokes formula on $[0,1]^n$ holds for $v$ whereas, for $1 \leqslant i \leqslant n-1$,

$$\int_{[0,1]^{n-1}} (w_i(x',1) - w_i(x',0))\partial_i\varphi(x')dx'$$
$$= (-1)^{n-1}\int_0^1\int_{[0,1]^{n-1}} \frac{\partial w_i(x',x_n)}{\partial x_n}\partial_i\varphi(x')dx'\wedge dx_n .$$

Consequently, the Stokes formula holds for $h^*u$ on $[0,1]^n$, so that

$$\int_{\partial K} u = \int_{\partial[0,1]^n} h^*u = \iint_{[0,1]^n} d(h^*u) = \iint_{[0,1]^n} h^*(du)$$
$$= \iint_{[0,1]^n} h^*f = \iint_K f$$

and the proof is complete. □

DEFINITION 1.5. *Let* $\Omega$ *be a Lipschitz domain in* $\mathbf{R}^n$. *An integrally continuous* $(n-1)$-*form* $u$ *on* $\Omega$ *is called absolutely continuous on* $\Omega$ *if* $d(u\,|\,_{\mathring{\Omega}})$, *taken in the distribution sense, is integrable on* $\mathring{K}$ *for any compact subset* $K$ *of* $\Omega$.

Note that if $u = \sum_{i=1}^n (-1)^{i-1}u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ and $u_i$ are, for instance, locally Lipschitz on $\Omega$, then $u$ is absolutely continuous on $\Omega$.

A simple consequence of Theorem 1.3 and of the above definition is the next.

THEOREM 1.6. *If* $K$ *is a compact Lipschitz domain in* $\mathbf{R}^n$ *and* $u$ *is an absolutely continuous* $(n-1)$-*form on* $K$, *then*

$$\int_{\partial K} u = \iint_{\mathring{K}} du .$$

## 2. CHARACTERIZATIONS OF THE PLURIDIMENSIONAL ABSOLUTE CONTINUITY

Theorem 1.3 suggests the possibility of characterizing pluridimensional absolute continuity of $(n-1)$-forms in a way similar to Lebesgue's definition of absolute continuity of functions on the real line (i.e. without involving the exterior derivative operator). This is made precise in the following theorem.