1. Three définitions of the first order Euler characteristic

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1. Three definitions of the first order Euler characteristic

Recall three definitions of the Euler characteristic, $\chi(X)$, of a finite complex X.

Definition A_0. $\chi(X) = \sum_{k \geqslant 0}(-1)^k$ (number of k-cells in X).

Definition B_0. $\chi(X; R) = \sum_{k \geqslant 0}(-1)^k \operatorname{rank}_R H_k(X; R)$ where R is a principal ideal domain. (This integer is independent of R.)

When X is an oriented manifold, M, we also have:

Definition C_0. $\chi(M)$ = intersection number of the graph of the identity map of M with itself.

We will introduce a higher analog called "the first order Euler characteristic" of X. There will be three analogous definitions, labelled A_1, B_1, and C_1 corresponding to the above definitions of the classical Euler characteristic. We prove in § 10 that under appropriate hypotheses these new definitions are equivalent.

First, we establish some notation. Let X be a finite connected CW complex with base vertex v. Write $G \equiv \pi_1(X, v)$ and $\Gamma \equiv \pi_1(X^X, \mathrm{id})$ where X^X is the function space of all continuous maps $X \to X$. Each $\gamma \in \Gamma$ can be represented by a cellular homotopy $F^\gamma: X \times I \to X$ such that $F_0^\gamma = F_1^\gamma = \mathrm{id}_X$. Orient the cells of X, thus establishing a preferred basis for the integral cellular chains $(C_*(X), \partial)$. Choose a lift, $\tilde{e}$, in the universal cover, $\tilde{X}$, for each cell e of X, and orient $\tilde{e}$ compatibly with e. Regard the cellular chain complex $(C_*(\tilde{X}), \tilde{\partial})$ as a free right $\mathbf{Z}G$-module chain complex with preferred basis $\{\tilde{e}\}$. Let $D_*^\gamma: C_*(X) \to C_{*+1}(X)$ be the chain homotopy induced by F^γ.

Sign Convention. If e is an oriented k-cell of X then $D_k(e)$ is the $(k+1)$-chain $(-1)^{k+1}F_*(e \times I) \in C_{k+1}(X)$, where $e \times I$ is given the product orientation.

Let R be a commutative ring. Regard ${}_R\tilde{\partial}_k \equiv \tilde{\partial}_k \otimes \mathrm{id}: C_k(\tilde{X}) \otimes R \to C_{k-1}(\tilde{X}) \otimes R$ and ${}_RD_k^\gamma \equiv D_k^\gamma \otimes \mathrm{id}: C_k(X) \otimes R \to C_{k+1}(X) \otimes R$ as matrices over RG and R respectively using the preferred bases. The abelianization homomorphism $A: G \to G_{ab} \cong H_1(X)$ extends to a homomorphism of R-modules $A: RG \to H_1(X; R) = H_1(X) \otimes R$.

We can now state the first definition of our *first order Euler characteristic* with coefficients in a commutative ring R. It is a homomorphism $\chi_1(X; R): \Gamma \to H_1(X; R)$. When $R = \mathbf{Z}$ we write, in abbreviated form, $\chi_1(X): \Gamma \to H_1(X)$. Note that Γ is abelian, and when X is aspherical, $\Gamma \cong Z(G)$, the center of G; see Proposition 1.3.

Definition A_1. Let R be a commutative ring of coefficients.

$$\chi_1(X;R)(\gamma) = \sum_{k \geqslant 0} (-1)^{k+1} A(\mathrm{trace}({}_R\tilde{\partial}_{k+1}\, {}_R D_k^{\gamma})) .$$

Here, we are multiplying RG-matrices by R-matrices to obtain RG-matrices. Note that $\chi_1(X;R)(\gamma) = \chi_1(X)(\gamma) \otimes 1$. We will show (Corollary 2.10) that this formula is independent of the various choices that have been made. Note that in order to know the right hand side, we must have information at the chain level, namely the matrices ${}_R\tilde{\partial}_{k+1}$ and ${}_R D_k^{\gamma}$. Definition A_1 is the "reduction" of a trace in 1-dimensional Hochschild homology; the corresponding trace (of the identity map) in 0-dimensional Hochschild homology "reduces" in the same way to Definition A_0; see §2 for more on this.

Our second definition requires the assumption that $H_*(X;R)$ be a free R-module where R is a principal ideal domain. This will be true, for example, if R is a field. For each $k \geqslant 0$, choose a basis $\{b_1^k, \ldots, b_{\beta_k}^k\}$ for $H_k(X;R)$. Let $\{\bar{b}_j^k\}$ be the corresponding dual basis for $H^k(X;R)$. Let $\Phi^{\gamma}: X \times S^1 \to X$ be the obvious quotient obtained from F^{γ}, above. By means of the Künneth formula, Φ^{γ} induces $\Phi_*^{\gamma}: H_k(X;R) \otimes H_1(S^1;R) \to H_{k+1}(X;R)$. Let $u \in H_1(S^1;R)$ be the generator which defines the usual orientation on S^1.

Definition B_1. Let R be a principal ideal domain. Suppose that $H_*(X;R)$ is a free R-module.

$$\chi_1(X;R)(\gamma) = \sum_{k \geqslant 0} (-1)^{k+1} \sum_j \bar{b}_j^k \cap \Phi_*^{\gamma}(b_j^k \otimes u)$$

where $\cap$ is the cap product in the sense of [D_2].

It is straightforward to show that the formula in Definition B_1 is independent of the choice of basis for $X_*(X;R)$.

Remark. Throughout this paper we use Dold's conventions [D_2] for cap and cup products. These conventions are the same as those of [MS] but differ from those of [Sp]. Writing $\cap'$ and $\cup'$ for the cap and cup products of [Sp], we have $x \cap y = (-1)^{|x|(|x|-|y|)} x \cap' y$ and $u \cup v = (-1)^{|u||v|} u \cup' v$ where "$|\,|$" denote the degree of a homology or cohomology class.

The above expression for $\chi_1(X;R)(\gamma)$ can also be written:

$$\chi_1(X;R)(\gamma) = \sum_{i=1}^{\beta_1} \sum_{k,j} (-1)^{k+1} \langle \bar{b}_i^1 \cup \bar{b}_j^k, \Phi_*^{\gamma}(b_j^k \otimes u)\rangle b_i^1$$

where $\langle \cdot , \cdot \rangle$ denotes the Kronecker pairing. A trace formula of this kind, for parametrized maps $X \times Y \to X$ was introduced by R.J. Knill in [Kn]. In order to know the right hand side in Definition B_1, we only need homological information about Φ^γ and cup product information about $H^*(X; R)$. The theory of [Kn] when applied to the identity map of X yields Definition B_0; hence the analogy with Definition B_1 (also see § 10).

Our third definition, Definition C_1 below, is an analog of the geometric Definition C_0 of $\chi(X)$. Let M be a compact oriented smooth (or PL) manifold with boundary. The *fixed point set* of F^γ is $\mathrm{Fix}(F^\gamma) \equiv \{(x, t) \mid F^\gamma(x, t) = x\}$, i.e. the coincidence set of F^γ and the projection $p: M \times I \to M$. As before, we form $\Phi^\gamma: M \times S^1 \to M$. We may perturb Φ^γ to a smooth (or PL) map Ψ^γ whose image misses ∂M and whose graph meets the graph of the projection p transversely. Then $\mathrm{Fix}(\Psi^\gamma) \equiv \{(x, t) \mid \Psi^\gamma(x, t) = x\}$ is a closed 1-manifold which naturally carries the "intersection orientation", using the order (graph of p, graph of Ψ^γ), as explained, for example, in [DG, § 8 and § 11] and [GN_1, § 6(A)]. This oriented 1-manifold defines an integral 1-cycle, $U(\gamma)$, in $X \times S^1$. The integral homology class determined by this cycle will be called the *intersection class*. If R is a commutative coefficient ring, let $\theta_R(\gamma) \in H_1(M; R)$ be the image of the homology class represented by the cycle $U(\gamma) \otimes 1$ under $p_*: H_1(M \times S^1; R) \to H_1(M; R)$. When $R = \mathbf{Z}$ we write $\theta_{\mathbf{Z}}(\gamma) = \theta(\gamma)$.

Definition C_1. Let R be a commutative ring of coefficients.

$$\chi_1(M; R)(\gamma) = -\theta_R(\gamma).$$

Definitions A_1, B_1 and C_1 define homomorphisms $\Gamma \to H_1(X; R)$ which are related as follows:

THEOREM 1.1 (Equivalence).

(i) *When* R *is a principal ideal domain and* $H_*(X; R)$ *is a free* R*-module, Definitions* A_1 *and* B_1 *agree;*

(ii) *when* X *is an oriented manifold and* R *is any commutative coefficient ring, Definitions* A_1 *and* C_1 *agree.*

The proof of Theorem 1.1 is deferred until § 10 so as not to interrupt the development of the χ_1-invariant. It is a technical proof, more or less independent of everything else in the paper.

Suppose that $h: X \to Y$ is homotopy equivalence where Y is a finite CW complex. Let $h^{-1}: Y \to X$ be a homotopy inverse for h. Then the map $h_\#: X^X \to Y^Y$ given by $f \mapsto hfh^{-1}$ is a homotopy equivalence. In particular, $h_{\#}$ induces an isomorphism $(h_{\#})_*: \Gamma \cong \Gamma' \equiv \pi_1(Y^Y, \mathrm{id})$. The assertion that $\chi_1(X; R)$ is a "homotopy invariant" means that the diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\chi_1(X;R)} & H_1(X;R) \\ {\scriptstyle (h_{\#})_*} \downarrow & & \downarrow {\scriptstyle h_*} \\ \Gamma' & \xrightarrow{\chi_1(Y;R)} & H_1(Y;R) \end{array}$$

is commutative. Note that the vertical arrows are isomorphisms.

THEOREM 1.2. $\chi_1(X; R)$ *is a homotopy invariant.*

For the proof, see Corollary 2.10. Theorem 1.2 allows us to extend the definition of $\chi_1(X; R)$ to any topological space X which is homotopy equivalent to a finite complex.

Let $\mathscr{E}(X) \subset X^X$ be the subset of self homotopy equivalences of X and $\mathscr{E}(X, v) \subset \mathscr{E}(X)$ consist of those homotopy equivalences which fix v. There is an *evaluation fibration* $\mathscr{E}(X, v) \hookrightarrow \mathscr{E}(X) \xrightarrow{\eta} X$, where $\eta(f) = f(v)$. The homotopy exact sequence of this fibration yields the exact sequence:

$$\pi_1(\mathscr{E}(X, v), \mathrm{id}) \to \Gamma \xrightarrow{\eta_{\#}} G \to \pi_0(\mathscr{E}(X, v)) \to \pi_0(\mathscr{E}(X))$$

where $\Gamma \equiv \pi_1(X^X, \mathrm{id}) = \pi_1(\mathscr{E}(X), \mathrm{id})$ and $G = \pi_1(X, v)$. The group $\mathscr{G}(X) \equiv \eta_{\#}(\Gamma)$ is called the *Gottlieb subgroup* of G.

Gottlieb showed ([Got, Theorem I.4]) that $\mathscr{G}(X)$ lies in the subgroup consisting of those elements of G which act trivially on $\pi_n(X, v)$, for all $n \geqslant 1$; in particular, $\mathscr{G}(X) \subset Z(G)$, the center of G. Indeed by elementary obstruction theory one obtains (see [Got]):

PROPOSITION 1.3. *If* X *is aspherical then* $\mathscr{G}(X) = Z(G)$ *and* $\eta_{\#}: \Gamma \to Z(G)$ *is an isomorphism.*

In view of this, we will often identify Γ with $Z(G)$ when X is aspherical. (The example of $X = S^2$ shows that the kernel of $\eta_{\#}: \Gamma \to \mathscr{G}(X)$ may be nontrivial when X not aspherical.)

A group G *is of type* $\mathscr{F}$ if there exists a $K(G, 1)$ which is a finite complex. By Theorem 1.2, the first order Euler characteristic is a homotopy invariant. In particular, applying these definitions to any finite $K(G, 1)$ complex we obtain the *first order Euler characteristic of the group* G *of type* $\mathscr{F}$. For any commutative ring G of coefficients, it is a homomorphism $\chi_1(G; R): Z(G) \to G_{\mathrm{ab}} \otimes R$.

PROPOSITION 1.4. *Let* G *be of type* $\mathcal{F}$. *If* $\chi(G) \neq 0$ *then* $\chi_1(G; R)$ *is trivial for any coefficient ring* R.

Proof. The center, $Z(G)$, is trivial, by [Got, Theorem IV.1]. Indeed, a short proof of this fact is included below as Proposition 2.4. $\square$

We end this section with the promised fourth definition of $\chi_1(X, R)$ in terms of the transfer maps of [BG], [D_3]. For $\gamma \in \Gamma$, consider $\Phi^\gamma : X \times S^1 \to X$ as above. This defines $\bar{\Phi}^\gamma : X \times S^1 \to X \times S^1$ by $\bar{\Phi}^\gamma(x, z) = (\Phi^\gamma(x, z), z)$ which is a fiber map with respect to the trivial fibration $X \to X \times S^1 \to S^1$. There is an associated S-map (the *transfer*) $\tau(\bar{\Phi}^\gamma) : \Sigma^\infty S^1_+ \to \Sigma^\infty(X \times S^1)_+$. Here, the subscript "$+$" indicates union with a disjoint basepoint and "Σ^∞" denotes the suspension spectrum of a space. The S-map $\tau(\bar{F})$ induces a homomorphism in homology $\tau(\bar{\Phi}^\gamma)_* : H_*(S^1; R) \to H_*(X \times S^1; R)$.

THEOREM 1.5. *Let R be a field. Then* $\chi_1(X; R) = -p_* \tau(\bar{\Phi}^\gamma)_*([S^1])$. $\square$

This is proved in § 10.

2. DISCUSSION OF DEFINITION A_1

To explain where Definition A_1 comes from, we must review some basic facts about Hochschild homology. Then we show that the formula in Definition A_1 is well-defined and homotopy invariant.

Let R be a commutative ground ring and let S be an associative R-algebra with unit. If M is an $S - S$ bimodule (i.e. a left and right S-module satisfying $(s_1 m) s_2 = s_1(m s_2)$ for all $m \in M$, and $s_1, s_2 \in S$), the *Hochschild chain complex* $\{C_*(S, M), d\}$ consists of $C_n(S, M) = S^{\otimes n} \otimes M$ where $S^{\otimes n}$ is the tensor product of n copies of S and

$$\begin{aligned} d(s_1 \otimes \cdots \otimes s_n \otimes m) &= s_2 \otimes \cdots \otimes s_n \otimes m s_1 \\ &\quad + \sum_{i=1}^{n-1} (-1)^i s_1 \otimes \cdots \otimes s_i s_{i+1} \otimes \cdots \otimes s_n \otimes m \\ &\quad + (-1)^n s_1 \otimes \cdots \otimes s_{n-1} \otimes s_n m \,. \end{aligned}$$

The tensor products are taken over R. The n-th homology of this complex is the n-th *Hochschild homology of* S *with coefficient bimodule* M. It is denoted by $HH_n(S, M)$. If $M = S$ with the standard $S - S$ bimodule structure then we write $HH_n(S)$ for $HH_n(S, M)$.