

Objektyp: **Group**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **39 (1993)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **22.09.2024**

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(to compare this inequality to the inequality as originally stated by Zimmert one has to apply the gamma duplication formula). He chose  $s = 2$  to obtain

$$\frac{R}{w} \geq 0.02 \cdot \exp(0.46r_1 + 0.1r_2).$$

Zimmert deduced his regulator bounds by an ingenious, but quite involved, investigation of certain analytic properties of the partial Dedekind zeta function associated to the class of principal ideals of  $K$ .

In this note we show that it is possible to deduce the above theorem by a simple estimate from a certain, almost obvious, monotonicity property of Hecke's theta function associated to the maximal order of  $K$  (see below). Moreover, we indicate below how this method of proof can be refined to yield exactly Zimmert's bounds. The technique of estimating which we apply is a sort of simple variation of a method which is developed in [F-S] to obtain lower bounds for  $L^p$ -norms of a certain class of functions. It was found during a careful analysis of Zimmert's method and reflects, though it looks much easier, still very much the spirit of Zimmert's original proof.

2. PROOF. Let  $|\cdot|_j$  for  $1 \leq j \leq r := r_1 + r_2$  denote the archimedean absolute values of  $K$ , let  $G$  denote the  $r$ -fold direct product of the multiplicative group of the positive reals  $\mathbf{R}_+$ , and let  $V$  denote the image in  $G$  of the units of  $K$  under the map

$$\eta \mapsto (\dots, |\eta|_j^{n_j}, \dots),$$

where  $n_j$  equals 1 or 2 accordingly as  $|\cdot|_j$  is real or complex. Denote by  $\delta$  the group homomorphism

$$\delta: G \rightarrow \mathbf{R}_+, \quad \delta((\dots, x_j, \dots)) = x_1 \cdots x_r.$$

Its kernel contains  $V$ , and by Dirichlet's unit theorem  $\ker \delta/V$  is compact. We can thus fix a Haar measure  $\mu$  on  $G/V$  by requiring

$$\int_{G/V} g \circ \delta d\mu = \frac{R}{w} \int_0^\infty g(t) \frac{dt}{t}$$

for any integrable function on  $\mathbf{R}_+$ . Let

$$Z(s) := \gamma(s) \sum_{\alpha \in \mathfrak{R}} |N_{K/\mathbf{Q}} \alpha|^{-s},$$

where  $\mathfrak{R}$  is a set of representatives for the non-zero elements of  $\mathfrak{O}$ , the ring

of integers in  $K$ , modulo units. According to Hecke [H] (and according to the choice of  $\mu$ ) one has, for  $\operatorname{Re}(s) > 1$ , the integral representation

$$Z(s) = \int_{G/V} \Theta \delta^s d\mu .$$

Here  $\Theta$  is a smooth, non-negative and  $V$ -invariant function on  $G$ , which is given by

$$\Theta(x) = \sum_{\substack{\alpha \in \mathfrak{D} \\ \alpha \neq 0}} \exp \left( - \sum_{j=1}^r |\alpha|_j^2 x_j^{2/n_j} \right) .$$

The main observation for the proof of the theorem is the

LEMMA. *The function  $(1 + \Theta)\delta$  is increasing in each argument.*

*Proof.* This follows from Hecke's theta formula [H, p. 165-166]

$$(1 + \Theta(x)) \delta(x) = \frac{\pi^{\frac{n}{2}} 2^{r_2}}{\sqrt{|d|}} \sum_{\alpha \in \mathfrak{D}^{-1}} \exp \left( - \pi^2 \sum_{j=1}^r n_j^2 |\alpha|_j^2 x_j^{-2/n_j} \right) ,$$

i.e. by applying Poisson summation to the series defining  $\Theta(x)$  (here  $\mathfrak{D}$  and  $d$  denote the different and discriminant of  $K$ ).  $\square$

We can now give the

*Proof of the theorem.* For  $a \in \mathbf{R}_+$  set

$$I(a) := \int_{G/V} (1 + \Theta(x)) \delta(x) w((a\delta(x))^{s-1}) d\mu(x) ,$$

where we use

$$w(t) = t \max(0, \log(1/t)) ,$$

and where  $s > 1$  as in the theorem.

For any  $\varepsilon > 0$ , one has  $w(t) = O(t^\varepsilon)$  and  $|w(t+h) - w(t)| / |h| \leq |w'(t)| = O(t^{-\varepsilon})$  as  $t \rightarrow 0$ . Thus, using the convergence of the integral representation of  $Z(s)$  for  $s > 1$ , we deduce that the integral defining  $I(a)$  is finite, and, on applying Lebesgue's theorem, that  $I(a)$  is differentiable and its derivative is obtained by differentiating under the integral sign. Here we agree to use  $w'(1)$  for the derivative on the right, i.e.  $w'(1) = 0$ .

On replacing  $x$  by  $x/a^{1/r}$  in the integral defining  $I(a)$  we deduce from the lemma that  $I(a)$  is decreasing. Hence  $I'(a) \leq 0$ , from which we obtain, writing  $\sigma = s - 1$ ,

$$\begin{aligned} & -\frac{d}{da} \int_{G/V} \delta w((a\delta)^\sigma) d\mu \geq \frac{d}{da} \int_{G/V} \Theta \delta w((a\delta)^\sigma) d\mu \\ & = \sigma a^{\sigma-1} \int_{G/V} \Theta \delta^s w'((a\delta)^\sigma) d\mu \geq -\sigma a^{\sigma-1} \int_{G/V} \Theta \delta^s (1 + \log(a\delta)^\sigma) d\mu \\ & = -\sigma^2 a^{\sigma-1} Z(s) \left( \frac{1}{\sigma} + \log a + \frac{Z'(s)}{Z} \right). \end{aligned}$$

By the choice of  $\mu$  the left-hand side equals  $R\sigma/(ws^2a^2)$ . Multiplying the above inequality by  $s^2a^2/\sigma$  and then maximizing the right hand side, i.e. choosing

$$\frac{1}{\sigma} + \log a + \frac{Z'}{Z}(s) = -\frac{1}{s},$$

we find

$$(1) \quad \frac{R}{w} \geq \frac{s(s-1)}{e} \exp\left(-\frac{s}{s-1}\right) Z(s) \exp\left(-s \frac{Z'}{Z}(s)\right).$$

Finally,  $Z(s) \geq \gamma(s)$ , since the Dirichlet series  $D(s)$  in the definition of  $Z(s)$  satisfies  $D(s) > 1$ , and  $\frac{Z'}{Z}(s) \leq \frac{\gamma'}{\gamma}(s)$ , since  $D'(s) < 0$ . Thus, (1) implies the claimed inequality.  $\square$

3. CONCLUDING REMARKS. To obtain a lower bound as sharp as Zimmert's one can proceed as above, but with a variant  $\Theta_1$  of the function  $\Theta$ . Namely, fix a real number  $s > 1$ , and define  $\Theta_1(x)$  by the same series as  $\Theta(x)$  but with the term

$$\exp\left(-\sum |\alpha|_j x_j^{2/n_j}\right)$$

replaced by

$$\prod_{j=1}^r f_j(|\alpha|_j^{n_j} x_j),$$