## §2. The Maass space

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only on the discriminant $D:=r^{2}-4 m n$ and the residue class $r(\bmod 2 m)$. The Petersson scalar product on $J_{k, m}^{\text {cusp }}$ is normalized by

$$
\begin{gathered}
\langle\phi, \psi\rangle=\int_{\Gamma_{1}^{J} \backslash \mathscr{H} \times \mathbf{C}} \phi(\tau, z) \overline{\psi(\tau, z)} \exp \left(-4 \pi m y^{2} / v\right) v^{k-3} d u d v d x d y \\
(\tau=u+i v, z=x+i y) .
\end{gathered}
$$

For basic facts about Jacobi forms we refer to [9].

## §2. The MaAss SPace

### 2.1. Results

Let $F$ be a Siegel modular form of integral weight $k$ on $\Gamma_{2}$ and write the Fourier expansion of $F$ in the form

$$
F(Z)=\sum_{m \geqslant 0} \phi_{m}(\tau, z) e^{2 \pi i m \tau^{\prime}} \quad\left(Z=\left(\begin{array}{cc}
\tau & z  \tag{1}\\
z & \tau^{\prime}
\end{array}\right) \in \mathscr{H}_{2}\right) .
$$

Using the injection

$$
\Gamma_{1}^{J} \rightarrow \Gamma_{2}, \quad\left(\left(\begin{array}{ll}
a & b  \tag{2}\\
c & d
\end{array}\right),((\lambda, \mu), \kappa)\right) \mapsto\left(\begin{array}{cccc}
a & 0 & b & \mu \\
\lambda^{\prime} & 1 & \mu^{\prime} & \kappa \\
c & 0 & d & -\lambda \\
0 & 0 & 0 & 1
\end{array}\right)
$$

where $\left(\lambda^{\prime}, \mu^{\prime}\right)=(\lambda, \mu)\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$, and the transformation formula of $F$ it is easy to see that the functions $\phi_{m}$ are in $J_{k, m}$. The expansion (1) is referred to as the Fourier-Jacobi expansion of $F$.

Thus for any $m \in \mathbf{N}_{0}$ we obtain a linear map

$$
\begin{equation*}
\rho_{m}: M_{k}\left(\Gamma_{2}\right) \rightarrow J_{k, m}, F \mapsto \phi_{m} . \tag{3}
\end{equation*}
$$

Note that $\rho_{0}$ is equal to the Siegel $\Phi$-operator.
We shall be interested in the case $m=1$. For $k$ odd, $\rho_{1}$ is the zero map; in fact, any Jacobi form of odd weight and index one must vanish identically as is easily seen.

For $k$ even, $\rho_{1}$ was studied in detail by Maass $[28,29]$ who showed the existence of a natural map $V: J_{k, 1} \rightarrow M_{k}\left(\Gamma_{2}\right)$ such that the composite $\rho_{1} \circ V$ is the identity. More precisely, let $\phi \in J_{k, 1}$ with Fourier coefficients $c(n, r)$ ( $n, r \in \mathbf{Z} ; r^{2} \leqslant 4 n$ ) and for $m \in \mathbf{N}_{0}$ define

$$
\begin{equation*}
\left(V_{m} \phi\right)(\tau, z):=\sum_{n, r \in \mathbb{Z}, r^{2} \leqslant 4 m n}\left(\sum_{d \mid(n, r, m)} d^{k-1} c\left(\frac{m n}{d^{2}}, \frac{r}{d}\right)\right) q^{n} \zeta^{r} \tag{4}
\end{equation*}
$$

(if $m=0$, the term $\sum_{d \mid 0} d^{k-1} c(0,0)$ on the right of (4) has to be interpreted as $\frac{1}{2} \zeta(1-k)$; note that $\left.V_{1} \phi=\phi\right)$. Using a more invariant definition of $V_{m}$ in terms of the action of a set of representatives for $\Gamma_{1} \backslash\left\{M \in \mathbf{Z}^{(2,2)} \mid \operatorname{det} M=m\right\}$ one checks that $V_{m} \phi \in J_{k, m}[9, \S 4]$. Put

$$
(V \phi)(Z):=\sum_{m \geqslant 0}\left(V_{m} \phi\right)(\tau, z) e^{2 \pi i m \tau^{\prime}} \quad\left(Z=\left(\begin{array}{cc}
\tau & z \\
z & \tau^{\prime}
\end{array}\right) \in \mathscr{H}_{2}\right) .
$$

We denote by $T_{n}(n \in \mathbf{N})$ the usual Hecke operators on $M_{k}\left(\Gamma_{2}\right)$ resp. $S_{k}\left(\Gamma_{2}\right)$ [12, IV; $1, \mathrm{II}]$; thus, if $p$ is a prime, $T_{p}$ resp. $T_{p^{2}}$ correspond to the two generators

$$
\Gamma_{2}\left(\begin{array}{cc}
1_{2} & 0 \\
0 & p 1_{2}
\end{array}\right) \Gamma_{2} \text { resp. } \Gamma_{2} \operatorname{diag}\left(1, p, p^{2}, p\right) \Gamma_{2}
$$

of the local Hecke algebra of $\Gamma_{2}$ at $p$. We denote by $T_{J, n}(n \in \mathbf{N})$ the Hecke operators on $J_{k, m}$ resp. $J_{k, m}^{\text {cusp }}[9, \S 4]$.

Theorem 1. (Maass [28, 29], Andrianov [2]). Suppose that $k$ is even. The map $\phi \mapsto V \phi$ gives an injection $J_{k, 1} \rightarrow M_{k}\left(\Gamma_{2}\right)$ which sends cusp forms to cusp forms and is compatible with the action of Hecke operators. If $p$ is a prime, one has $T_{p} \circ V=V \circ\left(T_{J, p}+p^{k-2}(p+1)\right)$ and $T_{p^{2}} \circ V=V \circ\left(T_{J, p}^{2}+p^{k-2}(p+1) T_{J, p}+p^{2 k-2}\right)$.

The image of $J_{k, 1}$ under $V$ is called the Maass space and will be denoted by $M_{k}^{*}\left(\Gamma_{2}\right)$. One knows that $M_{k}^{*}\left(\Gamma_{2}\right)=\mathbf{C} E_{k}^{(2)} \oplus S_{k}^{*}\left(\Gamma_{2}\right)$ where $E_{k}^{(2)}$ is the Siegel-Eisenstein series of weight $k$ on $\Gamma_{2}$ and $S_{k}^{*}\left(\Gamma_{2}\right):=M_{k}^{*}\left(\Gamma_{2}\right) \cap S_{k}\left(\Gamma_{2}\right)$. Observe that $\operatorname{dim} M_{k}^{*}\left(\Gamma_{2}\right)=\operatorname{dim} J_{k, 1}$ grows linearly in $k$ while $\operatorname{dim} M_{k}\left(\Gamma_{2}\right)$ grows like $k^{3}$.

Note that Theorem 1 implies that $M_{k}^{*}\left(\Gamma_{2}\right)$ is stable under all Hecke operators and that it is annihilated by the operator

$$
\begin{equation*}
\mathscr{C}_{p}:=T_{p}^{2}-p^{k-2}(p+1) T_{p}-T_{p^{2}}+p^{2 k-2}, \tag{5}
\end{equation*}
$$

for every prime $p$.
Let $F \in M_{k}\left(\Gamma_{2}\right)$ be a non-zero Hecke eigenform and denote by $\lambda_{n}(n \in \mathbf{N})$ its eigenvalues under $T_{n}$. If $p$ is a prime, we put

$$
Z_{F, p}(X):=1-\lambda_{p} X+\left(\lambda_{p}^{2}-\lambda_{p^{2}}-p^{2 k-4}\right) X^{2}-\lambda_{p} p^{2 k-3} X^{3}+p^{4 k-6} X^{4}
$$

so that $Z_{F, p}\left(p^{-s}\right)(s \in \mathbf{C})$ is the local spinor zeta function of $F$ at $p$. We put

$$
Z_{F}(s):=\prod_{p} Z_{F, p}\left(p^{-s}\right) \quad(\operatorname{Re}(s) \gtrdot 0)
$$

One has

$$
Z_{F}(s)=\zeta(2 s-2 k+4)^{-1} \sum_{n \geqslant 1} \lambda_{n} n^{-s} \quad(\operatorname{Re}(s) \gg 0) .
$$

If $F$ is an Eisenstein series, then it is well-known that $Z_{F}(s)$ can be expressed in terms of products of Hecke $L$-functions of elliptic modular forms.

Suppose that $F$ is cuspidal. Then it was proved in [1, Chap. 3] that $Z_{F}(s)$ has a meromorphic continuation to $\mathbf{C}$ which is holomorphic everywhere if $k$ is odd and is holomorphic except for a possible simple pole at $s=k$ if $k$ is even. Moreover, the global function $Z_{F}^{*}(s):=(2 \pi)^{-s} \Gamma(s) \Gamma(s-k+2) Z_{F}(s)$ is $(-1)^{k}$-invariant under $s \mapsto 2 k-2-s$.

Let $M_{2 k-2}\left(\Gamma_{1}\right)$ be the space of modular forms of weight $2 k-2$ on $\Gamma_{1}$. Recall that a Hecke eigenform in $M_{2 k-2}\left(\Gamma_{1}\right)$ is called normalized if its first Fourier coefficient is equal to 1 .

Theorem 2 (Saito-Kurokawa conjecture; Andrianov [2], Maass [28, 29], Zagier [39]). Let $k$ be even and let $F$ be a non-zero Hecke eigenform in $M_{k}^{*}\left(\Gamma_{2}\right)$. Then there is a unique normalized Hecke eigenform $f$ in $M_{2 k-2}\left(\Gamma_{1}\right)$ such that

$$
Z_{F}(s)=\zeta(s-k+1) \zeta(s-k+2) L_{f}(s)
$$

where $L_{f}(s)$ is the Hecke L-function attached to $f$.
Theorem 2 in particular shows that $Z_{F}(s)$ has a pole at $s=k$ if $F$ is a Hecke eigenform in $S_{k}^{*}\left(\Gamma_{2}\right)$. The converse is also true as shown by Evdokimov [10] and Oda [31], i.e. the function $Z_{F}(s)$ is holomorphic everywhere if and only if $F$ lies in the orthogonal complement of $S_{k}^{*}\left(\Gamma_{2}\right)$.

Using Theorem 2 one can show that $M_{k}^{*}\left(\Gamma_{2}\right)=\underset{p}{\cap} \operatorname{ker} \mathscr{E}_{p}$ where $\mathscr{C}_{p}$ is defined by (5). Finally let us mention that Theorem 2 implies that a Hecke eigenform $F$ in $S_{k}^{*}\left(\Gamma_{2}\right)$ does not satisfy the generalized Ramanujan-Petersson conjecture which would require that $\lambda_{n}<_{\varepsilon, F} n^{k-3 / 2+\varepsilon}(\varepsilon>0)$.

The proof of Theorem 1 is based on the fact that the function $V \phi$, by definition, is symmetric w.r.t. $\tau$ and $\tau^{\prime}$ and that $\Gamma_{2}$ is generated by the matrix $\operatorname{diag}\left(\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)\right)$ (which acts on $\mathscr{H}_{2}$ by interchanging $\tau$ and $\tau^{\prime}$ ) and the image of $\Gamma_{1}^{J}$ under the map (2). For the compatibility statement of $V$ with Hecke operators one has to check the action of the latter on Fourier coeffi-
cients. The proof of Theorem 2 is based on a trace formula. We do not give here any more details. Good expositions can be found in [9] and [39].

### 2.2. Problems

i) Since for fixed $k$ the dimension of $J_{k, m}$ grows linearly in $m$, the map $\rho_{m}$ defined by (3) for $m>_{k} 0$ cannot be surjective. Is there any simple or nice description of the image of $\rho_{m}$ or $\left(i m \rho_{m} \mid S_{k}\left(\Gamma_{2}\right)\right)^{\perp}$ ? Let us mention here that one can express the Fourier-Jacobi coefficients of Poincaré series of exponential type on $\Gamma_{2}$ which generate $S_{k}\left(\Gamma_{2}\right)$, as certain infinite linear combinations of Poincaré series on $\Gamma_{1}^{J}$ [22]. Taking scalar products one obtains a characterization of $\left(i m \rho_{m} \mid S_{k}\left(\Gamma_{2}\right)\right)^{\perp}$ as the kernel of certain infinite systems of linear equations. This description, however, does not seem to be very illuminating (for example, it does not imply in any obvious way that $\rho_{1}$ is surjective).
ii) A skew-holomorphic Jacobi form of weight $k \in \mathbf{Z}$ and index $m \in \mathbf{N}_{0}$ on $\Gamma_{1}^{J}$ as introduced by Skoruppa is a complex-valued $C^{\infty}$-function $\phi(\tau, z)(\tau \in \mathscr{H}, z \in \mathbf{C})$ satisfying the following properties: 1$) \phi$ is holomorphic in $z$ and is annihilated by the heat operator $8 \pi i m \partial / \partial \tau-\partial^{2} / \partial z^{2}$; 2) $\phi$ satisfies the same transformation formula under $\Gamma_{1}^{J}$ as a holomorphic Jacobi form of weight $k$ and index $m$ (cf. §1.2) except that the factor $(c \tau+d)^{k}$ has to be replaced by $\left.(c \bar{\tau}+d)^{k-1}|c \tau+d| ; 3\right) \phi$ has a Fourier expansion of type

$$
\phi(\tau, z)=\sum_{n, r \in \mathbf{Z}, r^{2} \geqslant 4 m n} c(n, r) \exp \left(-\pi \frac{r^{2}-4 m n}{m} v\right) q^{n} \zeta^{r} \quad(v=\operatorname{Im}(\tau)) .
$$

Note that a skew-holomorphic Jacobi form of even weight and index 1 is identically zero as is easily seen.

Despite of the importance of skew-holomorphic Jacobi forms as demonstrated in $[34,36]$ it is not quite clear so far how they are related to Siegel modular forms. One difficulty, for example, is that if one starts with a real-analytic Siegel modular form of genus 2, the coefficients of the partial Fourier expansion of $F(Z)$ w.r.t. $e^{2 \pi i \tau^{\prime}}\left(\right.$ where as usual $\left.Z=\left(\begin{array}{cc}\tau & z \\ z & \tau^{\prime}\end{array}\right)\right)$ not only depend on $\tau$ and $z$ but also on $\operatorname{Im}\left(\tau^{\prime}\right)$, and it is a priori not obvious how to get rid of the latter variable and to produce "true" Jacobi forms.

Let $k$ be an odd integer and denote by $M_{1 / 2, k-1 / 2}\left(\Gamma_{2}\right)$ the space of SiegelMaass wave forms '" of type ( $1 / 2, k-1 / 2$ )'" as defined in [26], i.e. the space of real-analytic functions $F: \mathscr{H}_{2} \rightarrow \mathbf{C}$ which satisfy

$$
F(M<Z>)=\operatorname{det}(C \bar{Z}+D)^{k-1}|\operatorname{det}(C Z+D)| F(Z)
$$

for all $M=\left(\begin{array}{cc}\cdot & \cdot \\ C & D\end{array}\right) \in \Gamma_{2}$ and which are annihilated by the matrix differential operator

$$
\begin{gathered}
\Omega_{1 / 2, k-1 / 2}:=(Z-\bar{Z})\left((Z-\bar{Z}) \frac{\partial}{\partial Z}\right)^{\prime} \frac{\partial}{\partial \bar{Z}}+\frac{1}{2}(Z-\bar{Z}) \frac{\partial}{\partial \bar{Z}} \\
-\left(k-\frac{1}{2}\right)(Z-\bar{Z}) \frac{\partial}{\partial Z} \\
\text { where } \frac{\partial}{\partial Z}=\left(\begin{array}{cc}
\frac{\partial}{\partial \tau} & \frac{1}{2} \frac{\partial}{\partial z} \\
\frac{1}{2} \frac{\partial}{\partial z} & \frac{\partial}{\partial \tau^{\prime}}
\end{array}\right)
\end{gathered}
$$

and $\frac{\partial}{\partial \bar{Z}}$ is defined analogously (the notation "of type $(1 / 2, k-1 / 2)$ )" comes from the fact that the factor of automorphy of $F$ can be written as $\operatorname{det}(C \bar{Z}+D)^{k-1 / 2} \operatorname{det}(C Z+D)^{1 / 2}$ with appropriate choice of the square root).

Using certain invariance properties of $\Omega_{1 / 2, k-1 / 2}$ under the action of $\mathrm{Sp}_{2}(\mathbf{R})$ one can define Hecke operators $T_{n}(n \in \mathbf{N})$ on $M_{1 / 2, k-1 / 2}\left(\Gamma_{2}\right)$ in the usual way. Let

$$
E_{1 / 2, k-1 / 2}^{(2)}(Z):=\sum_{(C, D)} \operatorname{det}(C Z+D)^{-k+1}|\operatorname{det}((C Z+D))|^{-1}
$$

be the Maass-Siegel-Eisenstein series in $M_{1 / 2, k-1 / 2}\left(\Gamma_{2}\right)$ ([26;27,§18]; summation over all pairs ( $C, D$ ) of relatively prime symmetric ( 2,2 )-matrices inequivalent under left-multiplication by $G L_{2}(\mathbf{Z})$ ). Then the following can be shown:

1) The function $E_{1 / 2, k-1 / 2}^{(2)}$ is a Hecke eigenform whose spinor zeta function (defined in the same way as above) is equal to $\zeta(s-k+1)$ $\zeta(s-k+2) L_{E_{2 k-2}}(s)$ where $E_{2 k-2}$ is the normalized Eisenstein series of weight $2 k-2$ on $\Gamma_{1}$ (this implies that $E_{1 / 2, k-1 / 2}^{(2)}$ for all primes $p$ is annihilated by the Hecke operator $\mathscr{C}_{p}$ defined analogously as in (5));
2) if $e_{1 / 2, k-1 / 2 ; m}\left(\tau, z, \operatorname{Im}\left(\tau^{\prime}\right)\right)$ is the $m$-th Fourier-Jacobi coefficient of $E_{1 / 2, k-1 / 2}^{(2)}$ and if for $m>0$ one carries out a similar limit process as in [19, $\S 2$, Remark ii) after the proof of Thm. 1], i.e. essentially replaces $\operatorname{Im}\left(\tau^{\prime}\right)$ by $(\operatorname{Im}(z))^{2} / \operatorname{Im}(\tau)+\delta$ and lets $\delta \rightarrow \infty$, then one obtains a skewholomorphic Eisenstein series of weight $k$ and index $m$ (in fact, finite linear combinations of such Eisenstein series if $m$ is not squarefree).

The following questions therefore are suggestive:

1) if one starts with an arbitrary $F \in M_{1 / 2, k-1 / 2}\left(\Gamma_{2}\right)$, does the above limit process produce skew-holomorphic Jacobi forms of weight $k$ ?
2) define $M_{1 / 2, k-1 / 2}^{*}\left(\Gamma_{2}\right)$ as the subspace of $M_{1 / 2, k-1 / 2}\left(\Gamma_{2}\right)$ consisting of the intersection of the kernels of the operators $\mathscr{C}_{p}$ for all primes $p$. Does there exist a natural map $V$ from skew-holomorphic Jacobi forms of weight $k$ and index 1 to $M_{1 / 2, k-1 / 2}^{*}\left(\Gamma_{2}\right)$ similar as in the case of holomorphic Jacobi forms?

Recently, N.-P. Skoruppa [36] has developed a theory of theta lifts from skew-holomorphic Jacobi forms to automorphic forms on $\mathrm{Sp}_{2}$. It would be interesting to investigate if his lifts would provide (at least partial) answers to the above questions.
iii) So far a generalization of the Maass space to higher genus $n>2$ has not been given; in fact, in the general case it does not seem to be quite clear what one has to look for, except that (the cuspidal part) of a "Maass space" eventually should be generated by Hecke eigenforms which do not satisfy a generalized Ramanujan-Petersson conjecture. Note that there is a partial negative result by Ziegler [40, 4.2. Thm.] who showed by means of specific examples that for $n \geqslant 33$ the map which sends a Siegel modular form of weight 16 on $\Gamma_{n}:=\operatorname{Sp}_{n}(\mathbf{Z})$ to its first Fourier-Jacobi coefficient is not surjective.

On the other hand, there are very interesting numerical calculations for $n=3$ due to Miyawaki [30] which suggest that a Siegel-Hecke eigenform $F$ of even integral weight $k$ on $\Gamma_{3}$ could be constructed from a pair $(f, g)$ of elliptic Hecke eigenforms of weights $\left(k_{1}, k_{2}\right)$ equal to $(k, 2 k-4)$ or ( $k-2,2 k-2$ ) such that the (formal) spinor zeta function of $F$ should be equal to $L_{f}\left(s-k_{2} / 2\right) L_{f}\left(s-k_{2} / 2+1\right) L_{f \otimes g}(s)$ where $L_{f \otimes g}(s)$ essentially is the Rankin convolution of $f$ and $g$ ([loc. cit., §4]; note that for $n>2$ the analytic continuation of the spinor zeta function of a holomorphic Hecke eigenform on $\Gamma_{n}$ is not known).

## §3. SPINOR ZETA FUNCTIONS

### 3.1. Results

Although the Maass space $S_{k}^{*}\left(\Gamma_{2}\right)$ as discussed in the previous section is an important subspace of $S_{k}\left(\Gamma_{2}\right)$ in its own right, one quickly realizes that the "true" Siegel cusp forms on $\Gamma_{2}$ should lie in the orthogonal complement of $S_{k}^{*}\left(\Gamma_{2}\right)$ (cf. Theorem 2 in $\S 2$ and its discussion). Is is therefore even more

