

## 4. p-adic Mal'cev-Neumann fields

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4.  $p$ -ADIC MAL'CEV-NEUMANN FIELDS

To construct analogous examples of characteristic zero whose residue field has nonzero characteristic requires a more complicated construction. First we recall two results about complete discrete valuation rings. For proofs, see [17], pp. 32-34.

A valued field  $(F, \nu)$  is called *discrete* if  $\nu(F) = \mathbb{Z}$ .

PROPOSITION 1. *If  $R$  is a perfect field of characteristic  $p > 0$ , then there exists a unique field  $R'$  of characteristic 0 with a discrete valuation  $\nu$  such that the residue field is  $R$ ,  $\nu(p) = 1 \in \mathbb{Z}$ , and  $R'$  is complete with respect to  $\nu$ . (The valuation ring  $A$  of  $R'$  is called the ring of Witt vectors with coefficients in  $R$ .)*

For example, if  $R = \mathbb{F}_p$ , then  $R' = \mathbb{Q}_p$  with the  $p$ -adic valuation.

PROPOSITION 2. *Suppose  $F$  is field with a discrete valuation  $\nu$ , and  $t \in F$  satisfies  $\nu(t) = 1$ . Let  $S \subset F$  be a set of representatives for the residue classes with  $0 \in S$ . Then every element  $x \in F$  can be written uniquely as  $\sum_{m \in \mathbb{Z}} x_m t^m$ , where  $x_m \in S$  for each  $m$ , and  $x_m = 0$  for all sufficiently negative  $m$ . Conversely, if  $F$  is complete, every such series defines an element of  $F$ .*

Now for the construction. Let  $R$  be a perfect field of characteristic  $p$ , and let  $G$  be an ordered group containing  $\mathbb{Z}$  as a subgroup, or equivalently with a distinguished positive element. (When we eventually define our valuation  $\nu$ , this element 1  $\in G$  will be  $\nu(p)$ .) Let  $A$  be the valuation ring of the valued field  $(R', \nu')$  given by Proposition 1.

What we want is to have the indeterminate  $t$  stand for  $p$  in elements of  $A((G))$ , so we get elements of the form  $\sum_{g \in G} \alpha_g p^g$ . The problem is that some elements of  $A((G))$ , like  $-p + t^1$ , "should be" zero. So what we do is to take a quotient  $A((G))/N$  where  $N \subset A((G))$  is the ideal of elements that "should be" zero.

We say that  $\alpha = \sum_g \alpha_g t^g \in A((G))$  is a *null series* if for all  $g \in G$ ,  $\sum_{n \in \mathbb{Z}} \alpha_{g+n} p^n = 0$  in  $R'$ . (Recall that we fixed a copy of  $\mathbb{Z}$  in  $G$ .) Note that  $\alpha_{g+n} = 0$  for sufficiently negative  $n$ , since otherwise  $\text{Supp } \alpha$  would not be well-ordered. Also,  $\nu'(\alpha_{g+n} p^n) \geq n$ , so  $\sum_{n \in \mathbb{Z}} \alpha_{g+n} p^n$  always converges in  $R'$ . Let  $N$  be the set of null series.

PROPOSITION 3.  $N$  is an ideal of  $A((G))$ .

*Proof.* Clearly  $N$  is an additive subgroup. Let  $G' \subset G$  be a set of coset representatives for  $G/\mathbf{Z}$ . Suppose  $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$ ,  $\beta = \sum_{h \in G} \beta_h t^h \in N$ , and  $\alpha\beta = \sum_{j \in G} \gamma_j t^j$ . Then for each  $j \in G$ ,

$$\begin{aligned} \sum_{n \in \mathbf{Z}} \gamma_{j+n} p^n &= \sum_{\substack{g+h=j+n \\ n \in \mathbf{Z}}} \alpha_g \beta_h p^n \\ &= \sum_{\substack{h' \in G' \\ l, m \in \mathbf{Z}}} (\alpha_{j-h'+l} p^l) (\beta_{h'+m} p^m) \end{aligned}$$

(We write  $h = h' + m$  with  $h' \in G'$  and let  $l = n - m$ .)

Since  $\beta \in N$ ,  $\sum_{m \in \mathbf{Z}} \beta_{h'+m} p^m = 0$  for each  $h' \in G'$ , so we get  $\sum_{n \in \mathbf{Z}} \gamma_{j+n} p^n = 0$ . (These infinite series manipulations in  $R'$  are valid, because for each  $i \in \mathbf{Z}$ , only finitely many terms have valuation less than  $i$ , since each  $\gamma_{j+n}$  is a finite sum of products  $\alpha_g \beta_h$ .) Hence  $N$  is an ideal.  $\square$

Define the  $p$ -adic Mal'cev-Neumann field  $L$  as  $A((G))/N$ .

PROPOSITION 4. Let  $S \subset A$  be a set of representatives for the residue classes of  $A$ , with  $0 \in S$ . Then any element  $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$  is equivalent modulo  $N$  to a element  $\beta = \sum_{g \in G} \beta_g t^g$  with each  $\beta_g$  in  $S$ . Moreover,  $\beta$  is unique.

*Proof.* Let  $G' \subset G$  be a set of coset representatives for  $G/\mathbf{Z}$ . For each  $g \in G'$ , we may write

$$\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n = \sum_{n \in \mathbf{Z}} \beta_{g+n} p^n$$

with  $\beta_{g+n} \in S$ , by Proposition 2. (This is possible since  $R'$  is complete with respect to its discrete valuation.) Then  $\beta = \sum_{g \in G'} \sum_{n \in \mathbf{Z}} \beta_{g+n} t^n$  is a well-defined element of  $A((G))$ , since  $\text{Supp}(\beta) \subseteq (\text{Supp} \alpha) + \mathbf{N}$ , which is well-ordered by part 2 of Lemma 1. Finally  $\alpha - \beta \in N$ , by definition of  $N$ . The uniqueness follows from the uniqueness in Proposition 2.  $\square$

COROLLARY 3.  $L = A((G))/N$  is a field.

*Proof.* The previous proposition shows that any  $\alpha \in A((G))$  is equivalent modulo  $N$  to 0 or an element which is a unit in  $A((G))$  by Corollary 1.  $\square$

Proposition 4 allows us to write an element of  $L$  uniquely (and somewhat carelessly) as  $\beta = \sum_{g \in G} \beta_g p^g$ , with  $\beta_g \in S$ . Thus given  $S$ , we can speak of  $\text{Supp}(\beta)$  for  $\beta \in L$ . Define  $v: L \rightarrow G_\infty$  by  $v(\beta) = \min \text{Supp} \beta$ .

PROPOSITION 5. *The map  $\nu$  is a valuation on  $L$ , and is independent of the choice of  $S$ . The value group is  $G$  and the residue field is  $R$ .*

*Proof.* For  $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$ , define

$$w(\alpha) = \min_{g \in G} \left\{ g + \nu' \left( \sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n \right) \right\}.$$

The elements in the “min” belong to  $(\text{Supp}(\alpha) + \mathbf{N}) \cup \{\infty\}$ , which is well-ordered by part 2 of Lemma 1, so this is well defined. It's clearly unchanged if an element of  $N$  is added to  $\alpha$ . In particular, if we do so to get an element  $\alpha' \in A((G))$  with coefficients in  $S$ , we find  $w(\alpha) = w(\alpha') = \min \text{Supp } \alpha'$ . Thus if  $\beta$  is the image of  $\alpha$  in  $L$ ,  $\nu(\alpha) = w(\beta)$ . Since  $w$  is independent of the choice of  $S$ , so is  $\nu$ . If  $\alpha', \beta'$  are the representatives in  $A((G))$  with coefficients in  $S$  of elements  $\alpha, \beta \in L$ , then it is clear that  $w(\alpha'\beta') = w(\alpha') + w(\beta')$  (because the leading coefficient of  $\alpha'\beta'$  has valuation 0 under  $\nu'$ ) and that  $w(\alpha' + \beta') \geq \min\{w(\alpha'), w(\beta')\}$ . Thus  $\nu$  is a valuation.

The value group of  $\nu$  is all of  $G$ , since  $\nu(p^g) = g$  for any  $g \in G$ . The natural inclusion  $A \subset A((G))$  composed with the quotient map  $A((G)) \rightarrow L$  maps  $A$  into the valuation ring of  $L$ , which consists of series  $\sum_{g \geq 0} \alpha_g p^g$ , so it induces a map  $\phi$  from  $A$  to the residue field of  $L$ . The residue class of  $\sum_{g \geq 0} \alpha_g p^g$  equals  $\phi(\alpha_0) \in A$  (since the maximal ideal for  $L$  consists of series  $\sum_{g > 0} \alpha_g p^g$ ). Thus  $\phi$  is surjective. Its kernel is the maximal ideal of  $A$ , so  $\phi$  induces an isomorphism from the residue class field of  $A$  to that of  $L$ .  $\square$

For example, if  $R$  is any perfect field of characteristic  $p$ , and  $G = k^{-1}\mathbf{Z}$  for some  $k \geq 1$  (with its copy of  $\mathbf{Z}$  as a subgroup of index  $k$ ), then  $L = R'(\sqrt[k]{p})$  with the  $p$ -adic valuation.

LEMMA 3. *If  $\alpha = \sum_{g \in G} \alpha_g p^g$  and  $\beta = \sum_{g \in G} \beta_g p^g$  with  $\alpha_g, \beta_g \in S$  are two elements of  $L$ , then  $\nu(\alpha - \beta) = \min\{g \in G \mid \alpha_g \neq \beta_g\}$ . (The corresponding fact for the usual Mal'cev-Neumann fields is obvious.)*

*Proof.* Let  $w$  be the map used in the proof of the previous proposition. Let  $\alpha' = \sum_{g \in G} \alpha_g t^g$  and  $\beta' = \sum_{g \in G} \beta_g t^g$  in  $A((G))$ . Then  $\nu(\alpha - \beta) = w(\alpha' - \beta')$ . If  $g_0 = \min\{g \in G \mid \alpha_g \neq \beta_g\}$ , then the leading term of  $\alpha' - \beta'$  is  $(\alpha_{g_0} - \beta_{g_0})t^{g_0}$ , and the leading coefficient here has valuation 0 under  $\nu'$ , since  $\alpha_{g_0}, \beta_{g_0}$  represent distinct residue classes, so  $w(\alpha' - \beta') = g_0$ , as desired.  $\square$

*Remarks.* Since the construction of  $A$  from  $R$  is functorial (the Witt functor), it is clear that the construction of  $L$  from  $R$  is functorial as well (for

each  $G$ ). However, whereas the Witt functor is fully faithful on perfect fields of characteristic  $p$ , this new functor is not. For example, Proposition 11 (to be proved in Section 7) shows  $L$  can have many continuous (i.e. valuation-preserving) automorphisms not arising from automorphisms of  $R$ .

Our construction could be done starting from a non-abelian value group to produce  $p$ -adic Mal'cev-Neumann division rings, but we will not be interested in such objects.

## 5. MAXIMALITY OF MAL'CEV-NEUMANN FIELDS

A valued field  $(E, w)$  is an *immediate extension* of another valued field  $(F, v)$  if

- (1)  $E$  is a field extension of  $F$ , and  $w|_F = v$ .
- (2)  $(E, w)$  and  $(F, v)$  have the same value groups and residue fields.

A valued field  $(F, v)$  is *maximally complete* if it has no immediate extensions other than  $(F, v)$  itself. (These definitions are due to F.K. Schmidt, but were first published by Krull [8].) For example, an easy argument shows that any field  $F$  with the trivial valuation, or with a discrete valuation making it complete, is maximally complete.

**PROPOSITION 6.** *Let  $(F, v)$  be a maximally complete valued field with value group  $G$  and residue field  $R$ . Then*

- (1)  $F$  is complete.
- (2) If  $R$  is algebraically closed and  $G$  is divisible, then  $F$  is algebraically closed.

*Proof.* (1) The completion  $\hat{F}$  of  $F$  is an immediate extension of  $F$  (see Proposition 5 in Chapter VI, §5, no. 3 of [2]), so  $\hat{F} = F$ .

(2) The algebraic closure  $\bar{F}$  of  $F$  is in this case an immediate extension of  $F$  (see Proposition 6 in Chapter VI, §3, no. 3 and Proposition 1 in Chapter VI, §8, no. 1 of [2]), so  $\bar{F} = F$ .

(This delightful trick is due to MacLane [10].)  $\square$

**PROPOSITION 7.** *Any continuous endomorphism of a maximally complete field  $F$  which induces the identity on the residue field is automatically an automorphism (i.e., surjective).*

*Proof.* The field  $F$  is an immediate extension of the image of the endomorphism, which is maximally complete since it's isomorphic to  $F$ .  $\square$