2. BOUNDS AND LOCATIONS FOR ZEROS

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Since $z^{-1} \in W$ for all $z \in W$, it is sufficient to study $z \in W, |z| \leqslant 1$, and in some ways it is more natural to deal with the above power series.

Some of our methods and results are similar to those of Thierry Bousch [5, 6], whose work was brought to our attention by D. Zagier. The report [5] proves that the closure of the set of zeros of polynomials with coefficients 0, ± 1 is connected. The thesis [6] contains, along with a variety of other results, general methods for studying similar problems. In the area where our work overlaps [5, 6], we obtain a somewhat stronger result by proving path connectivity.

Boris Solomyak [16] has studied zeros of power series of the form (1.9), but with the c_k, $k \geqslant 1$, allowed to take any real values in the interval [0, 1]. He shows that the bound (2.4) holds there as well, and that there is a "spike" of real zeros along the negative real axis. However, the zeros of Solomyak's functions are substantially different from those we investigate. For example, he shows that segments of the boundary he investigates have everywhere dense sets of points where a tangent exists, as well as everywhere dense sets of points with no tangent. There are also no holes in Solomyak's set of zeros.

The paper of Brenti, Royle, and Wagner [7] discusses various properties of chromatic polynomials. While it is not directly related to our work, the numerical evidence it presents shows that zeros of chromatic polynomials may also exhibit fractal behavior. This may also be true for the partition function zeros of [3].

2. Bounds and Locations for Zeros

A polynomial $f(z) \in P$ can have multiple zeros. If $\zeta \neq 1$ is a d-th root of unity, then ζ is a zero of

$$g(z) = \sum_{j=0}^{d-1} z^j ,$$

and therefore a zero of $g(z^k)$ for any k such that $d \mid k - 1$. Hence it is a zero of multiplicity 2 for $g(z)g(z^k)$, a polynomial in P. Higher multiplicities can be obtained by iterating this procedure. On the other hand, we do not know whether any $z \in W$ that is not a root of unity can be a multiple root of any $f(z) \in P$. There do exist power series with coefficients 0, 1 that have double zeros z with $|z| < 1$, as will be shown in Section 3.

Inside a disk $\{z : |z| < r\}$ for $r < 1$, any polynomial $f(z) \in P$ can have only a bounded number of zeros. We prove a slightly more general result that will be used later on.

PROPOSITION 2.1. *Suppose that* $f(z)$ *is a power series of the form*

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k, \quad a_k = 0, 1 .$$

Then for any $r, 0 < r < 1, f(z)$ *has*

$$\leqslant 2(-\log(1 - r^{1/2}))(-\log r)^{-1} \tag{2.1}$$

zeros in $|z| \leqslant r$.

Proof. We apply Jensen's theorem (Theorem 3.61 of [17]). If $z_1, \ldots, z_n$ are the zeros in $|z| < R$, where $r < R < 1$, then we find that

$$\sum_{j=1}^{n} \log(R/|z_j|) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta , \tag{2.2}$$

since $f(0) = 1$. Therefore, if m is the number of zeros in $|z| < r$, we have

$$m(\log R - \log r) \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta . \tag{2.3}$$

Since

$$|f(Re^{i\theta})| \leqslant \sum_{k=0}^{\infty} R^k = (1 - R)^{-1} ,$$

we obtain

$$m \leqslant (-\log(1 - R))(\log R - \log r)^{-1} .$$

We now choose $R = r^{1/2}$, and this yields the bound (2.1). (Better bounds can be obtained by selecting R more carefully or estimating the integral of $\log|f(z)|$ in Jensen's theorem better.) □

We next consider bounds on the size of $z \in W$. Since $1/z \in W$ for $z \in W$, it suffices to consider $|z| \leqslant 1$.

THEOREM 2.1. *Suppose that* z *satisfies* $|z| < 1$ *and that* $f(z) = 0$ *for some power series of the form* (1.9). *Then*

$$\frac{|z|}{1 - |z|} \geqslant \left|\frac{2 - z}{1 - z}\right| , \tag{2.4}$$

and $|z| \geqslant \varphi^{-1}$, *with equality if and only if* $z = -\varphi^{-1}$ *and* $f(z) = 1 + z + z^3 + z^5 + \cdots$. *Furthermore, there exists a* $\delta > 0$ *such that if* $|z| < \varphi^{-1} + \delta$, *then* z *is a negative real number.*

Proof. We note that

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k = 1 + \frac{1}{2} \sum_{k=1}^{\infty} z^k + g(z)$$

$$= \frac{2-z}{2(1-z)} + g(z) \tag{2.5}$$

where

$$g(z) = \frac{1}{2} \sum_{k=1}^{\infty} \varepsilon_k z^k , \qquad \varepsilon_k = \pm 1 \text{ for all } k . \tag{2.6}$$

Now for $|z| < 1$,

$$|g(z)| \leqslant \frac{|z|}{2(1-|z|)} , \tag{2.7}$$

and so we conclude that if (2.4) is violated, then $f(z) \neq 0$. Equality is possible in (2.7) only if z is real, and we easily check that for $z \in (-1, 1)$,

$$\frac{|z|}{1-|z|} = \left| \frac{2-z}{1-z} \right|$$

only at $z = -\varphi^{-1}$. For $z = -\varphi^{-1}$, equality holds in (2.7) when $\varepsilon_k = (-1)^{k-1}$ for $k \geqslant 1$; i.e., when

$$f(x) = 1 + x + x^3 + x^5 \cdots .$$

Let $\Delta = \{z : |z| < \varphi^{-1}\}$. Then $z \mapsto (2-z)/(1-z)$ maps Δ to the interior of the circle one of whose diameters is

$$\left[\frac{2+\varphi^{-1}}{1+\varphi^{-1}} , \frac{2-\varphi^{-1}}{1-\varphi^{-1}} \right] = [\varphi, 2+\varphi]$$

but $z \mapsto |z|/(1-|z|)$ maps Δ to

$$\left[0, \frac{\varphi^{-1}}{1-\varphi^{-1}} \right) = [0, \varphi)$$

so (2.4) fails if $z \in \Delta$. Moreover, if $z \in \partial\Delta$, (2.4) still fails unless

$$\frac{2-z}{1-z} = \varphi$$

$$z = \frac{\varphi - 2}{\varphi - 1} = -\varphi^{-1} .$$

We next prove that the only $z \in \bar{W}$ with $|z|$ close to φ^{-1} are negative real numbers. Since $\bar{\Delta}$ intersects the closed set

$$\left\{z : |z| \leqslant 0.9 \text{ and } \frac{|z|}{1-|z|} \geqslant \left|\frac{2-z}{1-z}\right|\right\}$$

only at $z = -\varphi^{-1}$, there exist $\delta_1, \delta_2 \in (0, 10^{-10})$ such that (2.4) fails for z in

$$S = \{z : |z| \leqslant \varphi^{-1} + \delta_1, |z + \varphi^{-1}| \geqslant \delta_2\} . \tag{2.8}$$

It only remains to find the possible elements of $\bar{W}$ that lie in

$$T = \{z : |z + \varphi^{-1}| < \delta_2, |z| < \varphi^{-1} + \delta_1\} . \tag{2.9}$$

For $z \in T$,

$$\operatorname{Re}\left(1 + \frac{z}{2(1-z)}\right) \geqslant \varphi^{-1} - 10|z + \varphi^{-1}| \geqslant \varphi^{-1} - 10^{-9} , \tag{2.10}$$

so if $z \in W$, then we must have $\operatorname{Re} g(z) \leqslant -\varphi^{-1} + 10^{-9}$. Since $|g'(z)| \leqslant 10$ for $z \in T$, and $|g(-\varphi^{-1})| \leqslant \varphi^{-1}$, to achieve $\operatorname{Re} g(z) \leqslant -\varphi^{-1} + 10^{-9}$, we must have $\varepsilon_k = (-1)^{k-1}$ for $1 \leqslant k \leqslant 20$, say. Then

$$f(z) = 1 + z + z^3 + \cdots + z^{19} + h(z) , \tag{2.11}$$

where

$$h(z) = \sum_{k=21}^{\infty} a_k z^k , \qquad a_k = 0 \text{ or } 1 . \tag{2.12}$$

Hence for $|z| = r$,

$$|h(z)| \leqslant \frac{r^{21}}{1-r} ,$$

while

$$1 + z + z^3 + \cdots + z^{19} = 1 + z\frac{1-z^{20}}{1-z^2} = \frac{1+z-z^2-z^{21}}{1-z^2} . \tag{2.13}$$

On the circle $|z| = r = 0.7$,

$$|1 + z - z^2| = |z - \varphi||z + \varphi^{-1}| \geqslant 0.08 \cdot 0.9 \geqslant 0.07 ,$$

so

$$\left|\frac{1+z-z^2-z^{21}}{1-z^2}\right| \geqslant \frac{0.07 - (0.7)^{21}}{1 + (0.7)^2} \geqslant 0.03 . \tag{2.14}$$

On the other hand, $(0.7)^{21}/0.3 < 0.01$, so by Rouché's theorem $f(z)$ and $(1+z-z^2-z^{21})/(1-z^2)$ have the same number of zeros inside $|z| \leqslant 0.7$. By the earlier part of the argument, and another application of Rouché's theorem, $1+z-z^2$ and $1+z-z^2-z^{21}$ have the same number of zeros inside $|z| \leqslant 0.7$, namely one. Therefore $f(z)$ has exactly one zero inside $|z| \leqslant 0.7$, and since $f(z)$ has real coefficients, this zero has to be real. □

The argument presented above is inefficient, and shows only that some value of $\delta < 10^{-10}$ is allowable. With a little more care one could show by an extension of the method used above that $\delta = 0.7 - \varphi^{-1} = 0.081\ldots$ is allowable, so that any $z \in W$ with $|z| < 0.7$ is real. In Section 6 we present a variation of this method that uses machine computation instead of careful estimates to establish rigorously that $\delta = 0.7 - \varphi^{-1}$ is allowable. Numerical evidence suggests that the minimal value of $|z|$ over $z \in \overline{W}\backslash\mathbf{R}$ is about 0.734. The method of Section 6 can be used to obtain estimates for the minimal value of $|z|$ over $z \in \overline{W}\backslash\mathbf{R}$ that are as accurate as one desires.

By Proposition 3.1 of the next section, $(-1, -\varphi^{-1}] \subseteq \overline{W}$. Since $\overline{W}$ is stable under $z \mapsto 1/z$ and closed, it follows that $[-\varphi, -\varphi^{-1}] \subseteq \overline{W}$.

In [8] it was shown that $z \in W$ implies $\mathrm{Re}(z) < 3/2$. Theorem 2.1 immediately leads to the bound $\mathrm{Re}(z) < 1.22$ for $z \in W$. Numerical evidence suggests that $\mathrm{Re}(z) < 1.14$ for $z \in W$. There are $z \in W$ with $\mathrm{Re}(z) > 1.13$. The methods outlined in Section 6 can be used to obtain precise bounds.

We can analyze inequality (2.4) for z close to 1. We find that for $z = 1 - x + iy$ with x and y small, $x > 0$, if $|y| \leqslant x^{3/2}$ then (2.4) fails, so $z \notin W$. We next show that there are points in W which approach 1 along trajectories tangent to the real axis.

PROPOSITION 2.2. *There exists a sequence of points* $z_n \in W$ *such that* $z_n \to 1$ *as* $n \to \infty$ *and*

$$|\mathrm{Im}(z_n - 1)| = o(\mathrm{Re}(z_n - 1)) \quad \textit{as} \quad n \to \infty . \tag{2.15}$$

Proof. Consider the polynomial

$$f_{m,n}(z) = 1 + z + \cdots + z^{m-1} + z^n \tag{2.16}$$

with $m \leqslant n$. For n large compared to m, we will show that $f_{m,n}(z)$ has a zero near to

$$\alpha = \alpha_{m,n} = \exp(\pi i/n + (\log m)/n) , \tag{2.17}$$

and $\mathrm{Re}(\alpha - 1) \sim (\log m)\,\mathrm{Im}(\alpha)$. We show that one can take $m \lesssim n/(\log n)$.

To show that $f_{m,n}(z)$ has a zero β near $\alpha = \alpha_{m,n}$, let

$$g(z) = m + z^n . \tag{2.18}$$

Then $g(\alpha) = 0$. Consider the circle $|z - \alpha| = (10n)^{-1}$. On this circle, $|g(z)| \geqslant m/100$, while

$$|(1 + z + \cdots + z^{m-1}) - m| \leqslant \sum_{k=1}^{m-1} |z^k - 1| = O(m^2/n) , \tag{2.19}$$

so for $m = o(n)$, by Rouché's theorem $g(z)$ and $f_{m,n}(z)$ have the same number of zeros inside the circle, namely one. This proves the claim and answers the Conway-Parker question. □

3. A NEIGHBORHOOD OF THE UNIT CIRCLE

In this section we prove that an open neighborhood of $\{z : |z| = 1, z \neq 1\}$ is contained in $\bar{W}$.

LEMMA 3.1. *If* $B \subseteq \mathbf{C}$ *is compact,* $n \geqslant 1, |z| < 1,$ *and*

$$B \subseteq \bigcup_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{0,1\}} \left[\left(\sum_{i=1}^{n} \varepsilon_i z^i\right) + z^n B\right] , \tag{3.1}$$

then every element of B *is expressible in the form*

$$\sum_{i=1}^{\infty} \varepsilon_i z^i, \quad \varepsilon_i \in \{0, 1\} . \tag{3.2}$$

In particular, if $-1 \in B$, *then* $z \in \bar{W}$.

Proof. Given $b_m \in B$, inductively pick $b_{m+1} \in B$ and $\varepsilon_{mi} \in \{0, 1\}$, $m \geqslant 0$, $1 \leqslant i \leqslant n$ such that

$$b_m = \left(\sum_{i=1}^{n} \varepsilon_{mi} z^i\right) + z^n b_{m+1} .$$

Successive substitution yields

$$b_0 = \left(\sum_{m=0}^{M-1} \sum_{i=1}^{n} \varepsilon_{mi} z^{mn+i}\right) + z^{Mn} b_M .$$