III. Lifting to C^3

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **39 (1993)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **19.03.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

of C belongs to the interior of the polynomial hull of $\bar{W}$, then C is entirely included in the interior of the polynomial hull of $\bar{W}$.

By holomorphic curve we will mean a connected 1-dimensional holomorphic manifold.

Proof. Let O be the interior of the polynomial hull of $\bar{W}$. It has to be shown that the set of points $p \in C$ which belong to O is closed in C. It is obviously open. Things being so localized one has to face the following situation: a "small" analytic disk given by a holomorphic parametrization $\varphi: \bar{\Delta} \to C$ (Δ the unit disk in $\mathbf{C}$) so that $\varphi(1) \in O$, U^+ a side of M included in W (at least one of the two sides is such) hence in O, in some neighborhood of $\varphi(\bar{\Delta})$; and one has to show that $\varphi(0) \in O$. Fix ψ a holomorphic map from $\mathbf{C}$ into $\mathbf{C}^n$ so that: $\psi(e^{i\theta}) \simeq -\vec{N}$ for θ outside some small neighborhood of 0 (mod 2π), where $\vec{N}$ is the unit outer normal to M (with respect to U^+), at say the point $\varphi(0)$, and $\psi(0)$ is arbitrarily chosen.

For $\eta > 0$, η small enough $\varphi(e^{i\theta}) + \eta\psi(e^{i\theta}) \in O$ for all θ, hence $\varphi(0) + \eta\psi(0) \in O$. Taking into account some uniformity with respect to $\psi(0)$, this gives Lemma 2.

III. Lifting to $\mathbf{C}^3$

We are simply going to consider sets K in $\mathbf{C}^3$ rotationally invariant in the first variable, that we describe as follows. For each $t \in [0, t_0]$ we are given a compact set $K_t \subset \mathbf{C}^2$. We consider the set $K \subset \mathbf{C}^3$ which is the closure of the set $\{(w, z_1, z_2) \in \mathbf{C}^3; (z_1, z_2) \in K_{|w|}, |w| \leqslant t_0\}$. i.e.

$$K = \overline{\bigcup_{|w| \leqslant t_0} \{w\} \times K_{|w|}}.$$

$\hat{K}$ denotes the polynomial hull of K in $\mathbf{C}^3$, while $\widehat{\cup K_t}$ denotes the polynomial hull in $\mathbf{C}^2$ of the closure of the set $\bigcup_{t \leqslant t_0} K_t$.

Lemma 3. *Let $(0, \zeta_1, \zeta_2) \in \mathbf{C}^3$, the following are equivalent:*

$$\begin{cases} \text{(i)} & (0, \zeta_1, \zeta_2) \in \hat{K} \\ \text{(ii)} & (\zeta_1, \zeta_2) \in \widehat{\cup K_t}. \end{cases}$$

Proof. (i) $\Rightarrow$ (ii) is trivial. We are interested in (ii) $\Rightarrow$ (i). Let $P(w, z_1, z_2)$ be a polynomial in 3 variables. To P we associate the polynomial $\tilde{P}$ defined by

$$\tilde{P}(w, z_1, z_2) = P(0, z_1, z_2) = \frac{1}{2\pi} \int_0^{2\pi} P(e^{i\theta} w, z_1, z_2) d\theta.$$

Since K is invariant under rotation in the w variable:

$$\sup_K |\tilde{P}| \leqslant \sup_K |P| .$$

Set $Q(z_1, z_2) = P(0, z_1, z_2)$. Using (ii) one gets

$$|P(0, \zeta_1, \zeta_2)| = |Q(\zeta_1, \zeta_2)| \leqslant \sup_{\cup K_t} |Q| = \sup_K |\tilde{P}| \leqslant \sup_K |P| .$$

So (i) is established.

Remark. There is another approach to Lemma 3, which may better "explain" the situation, and that we just sketch. If $\varphi: \Delta \to \mathbf{C}^2$ is a holomorphic disk (φ continuous on $\bar{\Delta}$, holomorphic on Δ) and T is a continuous map from $\mathbf{R}/2\pi\mathbf{Z}$ into $[0, t_0]$ so that $\varphi(e^{i\theta}) \in K_{T(\theta)} (\theta \in [0, 2\pi))$, then $\varphi(0) \in \widehat{\cup K_t}$. One sees that $(0, \varphi(0)) \in \hat{K}$ by considering holomorphic disks $(Q, \varphi): \Delta \to \mathbf{C} \times \mathbf{C}^2$, with $Q(0) = 0$ and $|Q(e^{i\theta})| \simeq T(\theta)$. Carrying this out in general may require the use of the fundamental theorem by Poletsky [6], which says that, in an appropriate sense, polynomial hulls are always explained by holomorphic disks.

IV. Trepeau's example

Here we describe a class of examples. Let χ be a smooth real valued function defined on $[0, 1]$, constant in no neighborhood of 0, and so that $\chi(0) = 0$, $|\chi| < 1$. In one of the versions of Trepreau's original example $\chi(t) = t$. Let $\mathcal{M}$ be the generic 4-dimensional manifold in $\mathbf{C}^3$, given by:

$$\mathcal{M} = \{(w, z_1, z_2) \in \mathbf{C}^3, |w| < 1, z_1 = s_1 + i\chi(|w|^2)s_2 ,$$
$$z_2 = s_2 - i\chi(|w|^2)s_1; (s_1, s_2) \in \mathbf{R}^2\} .$$

Notice that on $\mathcal{M}$, $z_1^2 + z_2^2$ is a real valued function, (on $\mathcal{M}$, $z_1^2 + z_2^2 \geqslant 0$), hence:

(*) *Any function which depends only on* $(z_1^2 + z_2^2)$ *is a CR function on* $\mathcal{M}$.

This already gives example of *CR* functions which cannot be holomorphically extended to any wedge. The existence of such functions is related to the fact that $\mathcal{M}$ is not "minimal" (in the sense of Tumanov), it contains $\mathbf{C} \times \{0\} \times \{0\}$ as a (nongeneric) *CR* manifold of same *CR* dimension (see [9], [2]).