# 3. Elliptic spaces

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It remains to deduce the inequality (2) from (2.3). If the inequality (2) holds for some power series $h(z)$ it will also hold for $h(z^k)$, at the cost of replacing $K$ by $K^{\frac{1}{2k}}$. By (2.3) we are thus reduced to showing that the power series

$$\sum_{i=0}^{\infty} q_i z^i = \prod_{i=0}^{\infty} (1 + z^i)$$

satisfies (2). But this is an immediate consequence of a theorem of Hardy and Ramanujan [10]. □

COROLLARY OF PROOF. *If $G$ satisfies the hypotheses of Theorem 2.1 (2) then for some $k \in \mathbf{N}$,*

$$G(z) \underset{c}{\geqslant} \prod_{i=1}^{\infty} [1 + (z^k)^i] . \quad \square$$

## 3. ELLIPTIC SPACES

In this section we establish the ellipticity of the spaces listed in the introduction.

*3.1. Finite simply connected H-spaces, X.*

Because $X$ is an $H$-space, $H_*(\Omega X; \mathbf{F}_p)$ is commutative, all $p$. Since it has finite depth [3; Theorem A] it is elliptic [7; Prop. 3.2]. Hence $X$ is elliptic.

*3.2. Simply connected homogeneous spaces, $G /\!/ H$.*

We may suppose that $G$ is simply connected, and hence elliptic by § 3. The fibration $G \to G/H \to BH$ loops to the fibration $\Omega G \to \Omega(G/H) \to H$ in which $\pi_1(H)$ acts trivially in $H_*(\Omega G; \mathbf{F}_p)$ [1; Lemma 5.1]. Thus we can use the Serre spectral sequence to deduce polynomial growth for $H_*(\Omega(G/H); \mathbf{F}_p)$ from the same property for $H_*(\Omega G; \mathbf{F}_p)$.

*3.3. Fibrations $F \to X \to B$ with F, B elliptic.*

Here all spaces are simply connected and we can apply the Serre spectral sequence to deduce that $H_*(X; \mathbf{Z})$ is concentrated in finitely many degrees, and finitely generated in each. Hence $X$ has the weak homotopy type of a finite $CW$ complex. Loop the fibration $F \to X \to B$ and use the fact that $H_*(\Omega F; \mathbf{F}_p)$ and $H_*(\Omega B; \mathbf{F}_p)$ grow polynomially to deduce the same property for $H_*(\Omega X; \mathbf{F}_p)$.

*3.4. Simply connected Poincaré complexes X with $H^*(X;\mathbf{F}_p)$ at most doubly generated.*

Suppose $p \neq 2$ and $H = H^*(X;\mathbf{F}_p)$ contains an element of odd degree. Then it has an odd generator $\alpha$. Using Poincaré duality it is easy to see that there are only three possibilities for the algebra $H$:

$$H = \Lambda\alpha \quad \text{or} \quad \Lambda\alpha \otimes \Lambda\beta \quad \text{or} \quad \Lambda\alpha \otimes \mathbf{F}_p[\beta]/\beta^k .$$

In each case a simple, classical computation [11] produces $\mathrm{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$ and shows that it grows polynomially. Since the Eilenberg-Moore spectral sequence converges from $\mathrm{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$ to $H^*(\Omega X;\mathbf{F}_p)$, $H^*(\Omega X;\mathbf{F}_p)$ also has this property.

In all other cases ($p = 2$ or $H$ concentrated in even degrees) $H$ is a commutative local ring in the classic sense. Because $H$ satisfies Poincaré duality it is a Gorenstein ring. Now a theorem of Wiebe [12; Korollar p. 268] asserts (because $H$ has at most two generators) that $H$ is a polynomial algebra divided by a regular sequence. It is thus easy (and classical [11]) to compute $\mathrm{Tor}^H(\mathbf{F}_p, \mathbf{F}_p)$, and deduce that it grows polynomially. Hence so does $H_*(\Omega X;\mathbf{F}_p)$.

*3.5. Simply connected Dupin hypersurfaces E in $S^{n+1}$.*

In [9; Table 2.1] are listed the possibilities for $H_*(E;\mathbf{Z})$. We divide these into three cases, using the notation of [9].

*Case (a): E has the same integral homology as $S^k$ or as $S^k \times S^l$.*

In this case Poincaré duality shows that $E$ has the same integral cohomology ring as $S^k$ or as $S^k \times S^l$, and we can apply 3.4.

*Case (b): E has the rational homotopy type of $A_3(2)$, $A_3(4)$, $A_3(8)$, $A_4(2)$ or $A_6(2)$.*

In these cases the calculations of [9; §6] show explicitly that the ring $H^*(E;\mathbf{Z})$ is torsion free and generated by two elements. Thus each $H^*(E;\mathbf{F}_p)$ is doubly generated, and we can apply Wiebe's result as in 3.4.

*Case (c): E has the integral homology of $S^k \times S^l \times S^{k+l}$, with $k < l$.*

We need, in this case, to recall from [9; §2] that there are linear sphere bundles

$$S^k \to E \overset{\pi_0}{\to} B \quad \text{and} \quad S^l \to E \overset{\pi_1}{\to} B_1$$

with $B_0, B_1$ simply connected focal submanifolds of $S^{n+1}$. Moreover if $D_0, D_1$ denote the corresponding disk bundles with boundary $E$ then $S^{n+1} = D_0 \underset{E}{\cup} D_1$.

Fix $p \geqslant 0$ and consider the Serre spectral sequence for the fibration $S^k \to E \to B_0$ with coefficients in $\mathbf{F}_p$. If this fails to collapse then $H^k(\pi_0): H^k(B_0; \mathbf{F}_p) \to H^k(E; \mathbf{F}_p)$ is surjective. Since $l > k$ it is always true that $H^k(\pi_1)$ is surjective. Choose classes $\alpha \in H^k(B_0; \mathbf{F}_p)$, $\beta \in H^k(B_1; \mathbf{F}_p)$ mapping to the same non-zero class in $H^k(E; \mathbf{F}_p)$. The Mayer-Vietoris sequence for the decomposition $S^{n+1} = D_0 \underset{E}{\cup} D_1$ then gives a class $\gamma \in H^k(S^{n+1}; \mathbf{F}_p)$ restricting to $\alpha$ and $\beta$, which is absurd.

Thus the spectral sequence for $S^k \to E \to B_0$ collapses and so $H_*(B_0; \mathbf{F}_p) \cong H_*(S^l \times S^{l+k}; \mathbf{F}_p)$. Using Poincaré duality for $B_0$ we see that $H^*(B_0; \mathbf{F}_p)$ and $H^*(S^l \times S^{l+k}; \mathbf{F}_p)$ are isomorphic as graded algebras. Thus $B_0$ is elliptic by 3.4 and $E$ is elliptic by 3.3.

*3.6. Simply connected closed manifolds $M$ with a smooth action by a compact Lie group $G$, having a simply connected codimension one orbit.*

Here we may assume $G$ is connected. Let the orbit be $G/K$, and convert the inclusion of $G/K$ into a fibration $F \to G/K \to M$. From [9; Table 1.5] we see that for any $p$, $\dim H_i(F; \mathbf{F}_p) \leqslant 2$, all $i$. Thus applying the Serre spectral sequence to the fibration $\Omega(G/K) \to \Omega M \to F$ and using 3.1 for $G/K$ we see that $H_*(\Omega M; \mathbf{F}_p)$ grows polynomially.

*3.7. Simply connected manifolds $M \# N$ with each of the rings $H^*(M; \mathbf{Z})$, $H^*(N; \mathbf{Z})$ generated by a single class.*

By Van Kampen's theorem both $M$ and $N$ are simply connected, and so their fundamental cohomology classes are not torsion. Since each ring is monogenic, $H^*(M; \mathbf{Z})$ and $H^*(N; \mathbf{Z})$ are torsion free. Thus $H^*(M; \mathbf{F}_p)$ and $H^*(N; \mathbf{F}_p)$ are also monogenic, and so $H^*(M \# N; \mathbf{F}_p)$ is doubly generated. Now apply 3.4.

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