3. The Metric Theory of Continued Fractions

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Hurwitz [150] showed, among other things, that $\mu(\theta) \geqslant \sqrt{5}$; furthermore, $\mu\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$. Perron [234] showed that if

$$\theta = [a_0, a_1, a_2, \ldots] ,$$

then

$$\mu(\theta) = \limsup_{i \to \infty} ([a_{i+1}, a_{i+2}, a_{i+3}, \ldots] + [0, a_i, a_{i-1}, \ldots, a_1]) .$$

From this it follows that $\mu(\theta) < \infty$ if and only if θ is of constant type.

The range of $\mu(\theta)$, as θ takes on all irrational values, is known as the *Lagrange spectrum* and has been extensively studied. We direct the reader to the work of Lagrange [178, pp. 26-27]; Markoff [203, 204]; Heawood [138]; Perron [235]; Vinogradov, Delone, and Fuks [295]; Freiman [111]; Kinney and Pitcher [166]; Berštein [29]; Davis and Kinney [78]; Cusick [59, 62]; Flahive [117]; Cusick and Mendès France [69]; Wilson [301]; Dietz [89]; Pavone [232]; Prasad [249]; and especially the books of Koksma [172] and Cusick and Flahive [67].

For more on approximation by rational numbers, see Cassels [52], Schmidt [272], Kraaikamp and Liardet [313], Larcher [312].

3. The Metric Theory of Continued Fractions

Recall that $\mathscr{E}$ denotes the set of real numbers in (0, 1) with bounded partial quotients.

While it is easy to see $\mathscr{E}$ has uncountably many elements, nevertheless "most" numbers do *not* have bounded partial quotients. More precisely, we have the following

THEOREM 2 (Borel-Bernstein). *$\mathscr{E}$ is a set of measure* 0.

The theorem is due to Borel [38]. The original proof was not complete, as discussed in Bernstein [27]; further details were provided in a later paper of Borel [39]. For other proofs, see Hardy and Wright [135, Thm. 196] or Khintchine [160]. Also see Dyson [96].

Here is a sketch of a more general theorem: first, let us equate probability with Lebesgue measure, and assume x is a real number in (0, 1). Then, expanding x as a continued fraction, we have

$$x = [0, a_1, a_2, a_3, \ldots] ,$$

and we can consider each $a_i = a_i(x)$ to be a function of x. Then it is not difficult to show that

$$\Pr[a_n(x) = k] = \Theta\left(\frac{1}{k^2}\right).$$

From this, it follows that

$$\Pr[a_n(x) \geqslant k] = \Theta\left(\frac{1}{k}\right).$$

If the random variables $a_i(x)$ were *independent*, it would follow from the Borel-Cantelli lemma that

$$\Pr[a_n(x) \geqslant b_n \text{ infinitely often}] = 1$$

if and only if $\sum_{n\geqslant 1} \frac{1}{b_n}$ diverges. Unfortunately, the $a_i(x)$ are not independent, but they are "almost" independent; with some additional work, the result can be shown.

Now taking, e.g., $b_n = n$, we see that for almost all x, we have $a_n(x) \geqslant n$ infinitely often, and hence $\mathscr{E}$ is a set of measure 0.

Theorem 2 is a simple result in the *metric theory of continued fractions,* which had its origins in an 1812 letter from Gauss to Laplace. Gauss essentially stated [116] that

$$\lim_{n\to\infty} \Pr[a_n(x) = k] = \log_2\left(1 + \frac{1}{k(k+2)}\right),$$

and this was proven by Kuzmin [174, 175] and Lévy [186], independently.

Actually, even more is known. For example, Khintchine [160, 162] proved that if $f(n)$ is a non-negative function that does not grow too quickly, then with probability 1 we have

$$\lim_{m\to\infty} \frac{1}{m} \sum_{1\leqslant k\leqslant m} f(a_k) = \sum_{r\geqslant 1} f(r) \log_2\left(1 + \frac{1}{r(r+2)}\right).$$

Now setting $f(i) = 1$ if $i = n$, and $f(i) = 0$ otherwise, we see that with probability 1, the fraction $\log_2\left(1 + \frac{1}{r(r+2)}\right)$ of the partial quotients in the continued fraction expansion of a real number x are equal to r.

Some early papers discussing the distribution of partial quotients include Gyldén [123, 124]; Brodén [45]; and Wiman [302].

For the classical metric theory of continued fractions, see (in addition to the papers mentioned above) Lévy [187, 188, 189, 191]; Khintchine [162, 163]; and Denjoy [83, 84, 85]. For more recent improvements, see Szüsz [289, 290]; de Vroedt [296]; Wirsing [303]; Rieger [261]; Babenko [12]; and Babenko and Jur'ev [13].

A more modern approach derives these results using powerful methods of ergodic theory. A good introduction is the book of Billingsley [31]. Other articles include Knopp [168]; Doeblin [91]; Ryll-Nardzewski [266]; Hartman, Marczewski, and Ryll-Nardzewski [137]; Hartman [136]; Lévy [190]; Rényi [257]; de Vroedt [297]; Stackelberg [283]; Šalát [267]; Philipp [239, 240, 241, 242, 243]; Philipp and Stackelberg [244]; and Galambos [112, 113, 114].

4. Continued Fractions for Algebraic Numbers

A major open problem is to determine if any algebraic numbers of degree > 2 are in $\mathscr{B}$. As Khintchine [164, 165, 160] has remarked,

> It is interesting to note that we do not, at the present time, know the continued-fraction expansion of a single algebraic number of degree higher than 2. We do not know, for example, whether the sets of elements [partial quotients] in such expansions are bounded or unbounded. In general, questions connected with the continued-fraction expansion of algebraic numbers of higher degree than the second are extremely difficult and have hardly been studied.

(The problem goes back at least to 1949, with the appearance of Khintchine's book [164]. The paragraph above most likely also appeared in the first (1936) edition of Khintchine's book, but I have not been able to verify this by examining a copy. I do not know any earlier explicit reference to the problem. A remark similar to Khintchine's was made by Delone in a foreword to a translation of Delone and Fadeev [82, p. iv].)

Khintchine's remark is still true today; there are only a few papers that have explicitly discussed the partial quotients of algebraic numbers of degree > 2. See, for example, Davenport[1]) [76]; Orevkov [229]; Pass [231]; Wolfskill [304]; Blinov and Rabinovich [34]; Bombieri and van der Poorten [37]; Dzenskevich and Shapiro [98]; and van der Poorten [247].

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[1]) Actually, Davenport's results apply to all irrational numbers, not just algebraic numbers. Also see Mendès France [206].