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EMMY NOETHER: HIGHLIGHTS OF HER LIFE AND WORK

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Emmy Noether was a towering figure in the evolution of abstract algebra. In fact, she was the moving spirit behind the abstract, axiomatic approach to algebra. She also had a singular personality which attracted a group of students and collaborators who spread the gospel of abstract algebra far and wide. I will first give a sketch of her life and then discuss some of her work, including her intellectual debts and her legacy.

A. HER LIFE

Emmy Noether was born in 1882 in Erlangen, the German university town of Klein's Erlangen Programme fame. The university was founded in 1743 and had among its mathematics faculty such luminaries as von Staudt, Klein, Gordan (the "king of invariants"), and Max Norther, the famous algebraic geometer and Emmy's father. Gordan was a contemporary and friend of Max Noether and a frequent visitor of the Noethers. Although Emmy showed little early interest in mathematics, the frequent mathematical conversations at the Noether household between Gordan and her father were an important part of the atmosphere in which she grew up. Gordan was later to become Emmy Noether's thesis advisor.

Emmy Noether came from an economically well-established household, and her childhood seems to have been happy. She liked dancing and took piano lessons (which she did not like). She was a friendly and likeable child. Between the ages of seven and fifteen she went to the "Municipal School for Higher Education of Daughters", where she studied English and French.[1]) In 1900, at the age of eighteen, she was certified as a teacher of both subjects in

[1]) There is no indication that she wanted to study mathematics or science, but these were, of course, not "feminine" subjects.

"Institutions for the Education and Instruction of Females". She chose, however, not to pursue this career and instead enrolled at the University of Erlangen.

Easier said than done in those days. As the famous German historian Heinrich von Treitschke put it in the early 1890s ([17], p. 17):

> Many sensible men these days are talking about surrendering our universities to the invasion of women, and thereby falsifying their entire character. This is a shameful display of moral weakness. They are only giving way to the noisy demands of the Press. The intellectual weakness of their position is unbelievable... The universities are surely more than mere institutions for teaching science and scholarship. The small universities offer the students a comradeship which in the freedom of its nature is of inestimable value for the building of a young man's character...

In 1898, two years before Emmy Noether entered the University of Erlangen, the Senate of the University declared that the admission of women students would "overthrow all academic order" ([24], p. 10). By 1900, however, the authorities relented and extended to women the *conditional* right to enrol in German universities.[1]) Individual professors had the right, which they often exercised, to deny women permission to attend their lectures. This meant that Emmy Noether (one of two women among 1,000 students at the university) had to choose her subjects and instructors with some care. In fact, she at first took courses in history and modern languages, but later she switched to mathematics — it is not clear exactly when and why. By 1904 she was *formally* able to register as a student at the University of Erlangen, now studying only mathematics. In 1908 she received her Ph. D. degree, summa cum laude, having written a thesis on invariants under Gordan.

Between 1908 and 1915 Emmy Noether worked *without compensation* at the University of Erlangen. "Working" meant doing research, attending meetings of the German Mathematical Society and giving presentations, and occasionally substituting at lectures for her ailing father.

Although Emmy Noether did not have a formal position during these seven years, they were, as noted, not idly spent; and they bore fruit. She had become an expert on invariant theory, to the point that in 1915 Hilbert and Klein invited her to Göttingen to help them with problems on differential invariants.

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[1]) Women were permitted to enrol at universities in the U.S. in 1853, in France in 1861, in England in 1878, and in Italy in 1885.

These proved important in connection with their work on mathematical aspects of the general theory of relativity.[1])

Emmy Noether's move to Göttingen was of singular importance. Göttingen was at that time considered the world center of mathematics. With Gauss, Dirichlet, and Riemann as former professors, and with the contemporary faculty including Klein, Hilbert, Landau, Minkowski, and Courant, and later Weyl, Bernays, and Neugebauer, Göttingen had become the "Mecca of Mathematics". The list of visitors reads like a "who's who" of the world of mathematics: van der Waerden from Holland, Olga Taussky and Köthe from Austria, Tagaki and Shoda from Japan, André Weil and Chevalley from France, O. Schmidt, Gelfond, Alexandrov, Kolmogorov, and Urysohn from the Soviet Union, Tsen from China, Kuratowski from Poland, MacLane, G. D. Birkhoff, Wiener, and Lefschetz from the United States, and Artin, Hasse, Brauer, Siegel, and von Neumann from various universities in Germany.

Emmy Noether thrived in these surroundings. The decade 1920-1930 was the decisive period of her mathematical life. This is when she made her ground-breaking contributions to algebra. (She was then in her forties[2])) She attracted students, co-workers, and visitors who vigorously developed the subject soon to become known as "modern algebra". About 1930, the algebraists around Noether had gained the reputation as the most active group at the Mathematical Institute of Göttingen — quite an accomplishment given the presence of the likes of Hilbert, Weyl, Landau, and Courant! Two great honours came her way in 1932. First, she was awarded, jointly with Artin, the "Ackermann-Teubner Memorial Prize" for the advancement of the mathematical sciences. Second, she gave one of the 21 plenary addresses at the International Congress of Mathematicians in Zurich.[3])

But all did not go smoothly for Emmy Noether at Göttingen. One began a university career in Germany as Privatdozent (comparable in rank to an assistant professor). This was an *unpaid* position which gave its holder the right to teach.[4]). The income of Privatdozenten consisted of minimal fees paid by students for attending their lectures. One would have thought that under

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[1]) It was 1915, and Einstein had just promulgated his general theory of relativity. Both Hilbert and Klein turned their attention to it.

[2]) "Such a late maturing is a rare phenomenon in mathematics", notes Weyl ([41], p. 128), mentioning Sophus Lie as another great exception to the rule.

[3]) It was a remarkable event for a woman to be invited to give a plenary talk.

[4]) The Privatdozent, unlike the professor, was not an appointee of the state and hence received no salary.

such circumstances, and having been invited to Göttingen by the great Klein and Hilbert, Emmy Noether would have got an appointment as Privatdozent immediately upon her arrival in Göttingen. That was not to be, however. The philologists and historians of the Philosophical Faculty (of which mathematics was part) opposed Hilbert's efforts to allow Emmy to habilitate (a necessary step in becoming Privatdozent), because she was a woman. Hilbert protested, without success, to the University Senate: "After all", he claimed, "we are a university, and not a bathing establishment" ([41], p. 125). Only four years later (in 1919) was Noether allowed to habilitate and become Privatdozent. This followed the war, which brought profound political and social change, including an improvement in the legal position of women.

Three years later, in 1922, the mathematics department of Göttingen applied to the Ministry of Education to appoint Noether as professor. She was given the title "extraordinary professor without tenure" ("extraordinary" is the equivalent of an associate professor). This was merely a title, carrying no obligations and no salary. Since the high postwar inflation in Germany greatly reduced students' ability to pay their instructors, Noether was fortunate to get in the following year a "Teaching Assignment" ("Lehrauftrag") in Algebra, which provided a small remuneration. It required, however, annual confirmation by the Ministry. This is the position she remained in until she left Göttingen ten years later.

Why was there little *institutional* recognition of Emmy Noether's talents and accomplishments? We can only speculate, of course. But she had several marks against her: she was a woman, she was Jewish, and she had leftist political sympathies.

What kind of teacher was Emmy Noether? By standard measures, she was not a good teacher. She did not give well-organized, polished lectures. Yet, she inspired many students, through her lectures and through personal contact. Here is testimony from some who attended her courses:

> She was concerned with concepts only, not with visualization or calculation... This... was probably one of the main reasons why her lectures were difficult to follow... And yet, how profound the impact of her lecturing was! (Van der Waerden [37], p. 110).

> Professor Noether's lectures... are... excellent, both in themselves and because they bear an entirely different character in their excellence. Professor Noether thinks fast and talks faster. As one listens, one must also think fast — and that is always excellent training. (MacLane [29], p. 77).

> To an outsider Emmy Noether seemed to lecture poorly, in a rapid and confusing manner, but her lectures contained a tremendous force of

mathematical thought and an extraordinary warmth and enthusiasm. (Alexandrov [2], p. 165).

Indeed, Emmy Noether had a warm and caring personality. She was also modest, generous, frank, strong-willed, and outwardly coarse. "She was both a loyal friend and a severe critic", said van der Waerden ([37], p. 111), giving expression to one of these seeming contradictions. Her personal traits, combined with deep mathematical insights, attracted a core of devoted students, the so-called "Noether boys".[1]) They often visited her home, and they used to go on frequent walks together. The topic of conversation was almost invariably mathematics. Here is the story of one such walk.

It was raining, and Emmy Noether's umbrella did not offer much protection since it was in poor condition. When her students suggested that she get it repaired, she replied: "Quite right, but it can't be done: when it doesn't rain, I don't think of the umbrella, and when it rains, I need it" ([12], p. 48). In a more serious vein, van der Waerden relates the following ([38], p. 173):

> I wrote a paper based upon this simple idea and showed it to Emmy Noether. She at once accepted it for the *Mathematische Annalen,* without telling me that she had presented the same idea in a course of lectures just before I came to Göttingen. I heard it later from Grell, who had attended her course.[2])

On January 31, 1933 Hitler assumed the office of Chancellor. On March 31 he announced the beginning of the Third Reich. On April 25 Emmy Noether was dismissed from her teaching position. The dismissal of Courant, Landau, and Bernays followed in short order. Courant was replaced as head of the Mathematics Institute at Göttingen by Neugebauer, who lasted one day in that position. He refused to sign the required loyalty declaration.

With Weyl's assistance, Emmy Noether got a visiting position at Bryn Mawr College in Pennsylvania. The transition might have been difficult but for the warm reception she received at Bryn Mawr and the mathematical contacts she established at nearby Princeton. At Bryn Mawr she had her "Noether girls" — one doctoral and three postdoctoral students (among the latter was Olga Taussky). At Princeton she began (in early 1934) to give weekly lectures on algebra. Writing to Hasse about them, she said: "I'm beginning to realize that I must be careful; after all, they are essentially used to explicit

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[1]) Among her Ph.D. students were Deuring, Fitting, Grell, Greta Hermann, Krull, Levitzki, F.K. Schmidt, Ruth Stauffer, and Witt.

[2]) Emmy Noether was a collaborator in the editing of *Mathematische Annalen,* but she was hurt that this work was never explicitly recognized. Grell was one of her Ph.D. students.

computation and I have already driven a few of them away with my approach'' ([12], pp. 81-82). Among those who were not driven away were Albert, Brauer, Jacobson, Vandiver, and Zariski. In a recent book on Zariski, Carol Parikh pointed out that ''Zariski's contact with Noether was undoubtedly the single most important aspect of that year for him'' ([33], p. 74).

The time she spent at Bryn Mawr and Princeton was the happiest in her life, Emmy Noether told Veblen before her death. She was respected and appreciated as she had never been in her own country. But it was a brief, if happy, year and a half. On April 10, 1935 she underwent an operation for a tumor. She was recovering well when, four days later, complications brought unexpected death.

Ten days after her death Hermann Weyl delivered at Bryn Mawr a moving and eloquent eulogy. Let me conclude this account of Emmy Noether's life by quoting from it ([41], pp. 132, 149-152; for further details about her life see [7], [12], [24], and [36]):

> It was only too easy for those who met her for the first time, or had no feeling for her creative power, to consider her queer and to make fun at her expense. She was heavy of build and loud of voice, and it was often not easy for one to get the floor in competition with her. She preached mightily, and not as the scribes. She was a rough and simple soul, but her heart was in the right place. Her frankness was never offensive in the least degree. In everyday life she was most unassuming and utterly unselfish; she had a kind and friendly nature. Nevertheless she enjoyed the recognition paid her; she could answer with a bashful smile like a young girl to whom one had whispered a compliment. No one could contend that the Graces had stood by her cradle; but if we in Göttingen often chaffingly referred to her as ''der Noether'' (with the masculine article), it was also done with a respectful recognition of her power as a creative thinker who seemed to have broken through the barrier of sex. She possessed a rare humor and a sense of sociability; a tea in her apartment could be most pleasurable... She was a kind-hearted and courageous being, ready to help, and capable of the deepest loyalty and affection. And of all I have known, she was certainly one of the happiest...
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> Two traits determined above all her nature: First, the native productive power of her mathematical genius. She was not clay, pressed by the artistic hands of God into a harmonious form, but rather a chunk of human primary rock into which he had blown his creative breath of life. Second, her heart knew no malice; she did not believe in evil – indeed it never entered her mind that it could play a role among men. This was never more

forcefully apparent to me than in the last stormy summer, that of 1933, which we spent together in Göttingen... A time of struggle like this one... draws people closer together; thus I have a particularly vivid recollection of these months. Emmy Noether, her courage, her frankness, her unconcern about her own fate, her conciliatory spirit, were in the midst of all the hatred and meanness, despair and sorrow surrounding us, a moral solace... The memory of her work in science and of her personality among her fellows will not soon pass away. She was a great mathematician, the greatest, I firmly believe, that her sex has ever produced, and a great woman.

B. Her work

I will now give an account of some of Emmy Noether's major contributions to mathematics, indicating their sources.

Irving Kaplansky called her the "mother of modern algebra" ([23], p. 155). Saunders MacLane asserted that "abstract algebra, as a conscious discipline, starts with Emmy Noether's 1921 paper 'Ideal Theory in Rings'" ([28], p. 10). Hermann Weyl claimed that she "changed the face of algebra by her work" ([41], p. 128). It is a tall order to try to do justice to these assertions, but let me try.

According to van der Waerden, the essence of Emmy Noether's mathematical credo is contained in the following maxim ([5], p. 42):

> All relations between numbers, functions and operations become perspicuous, capable of generalization, and truly fruitful after being detached from specific examples, and traced back to conceptual connections.

We identify these ideas with the abstract, axiomatic approach in mathematics. They sound commonplace to us. But they were not so in Emmy Noether's time. In fact, they are commonplace today in considerable part *because* of her work.

Algebra in the 19th century was concrete by our standards. It was connected in one way or another with real or complex numbers. For example, some of the great contributors to algebra in the 19th century, mathematicians whose works shaped the algebra of the 20th century, were Gauss, Galois, Jordan, Kronecker, Dedekind, and Hilbert. Their algebraic works dealt with quadratic forms, cyclotomy, field extensions, permutation groups, ideals in rings of integers of algebraic number fields, and invariant theory. All of these works were related in one way or another to real or complex numbers.

Moreover, even these important works in algebra were viewed in the 19th century, in the overall mathematical scheme, as secondary. The primary mathematical fields in that century were analysis (complex analysis, differential equations, real analysis), and geometry (projective, noneuclidean, differential, and algebraic). But after the work of Emmy Noether and others in the 1920s, algebra became central in mathematics.

It should be noted that Emmy Noether was not the only, nor even the only major, contributor to the abstract, axiomatic approach in algebra. Among her predecessors who contributed to the genre were Cayley and Frobenius in group theory, Dedekind in lattice theory, Weber and Steinitz in field theory, and Wedderburn and Dickson in the theory of hypercomplex systems. Among her contemporaries, Albert in the U.S. and Artin in Germany stand out.

The "big bang" theory rarely applies when dealing with the origin of mathematical ideas. So also in Emmy Noether's case. The concepts she introduced and the results she established must be viewed against the background of late-19th-and early-20th-century contributions to algebra. She was particularly influenced by the works of Dedekind. In discussing her contributions she frequently used to say, with characteristic modesty: "It can already be found in Dedekind's work" ("Es steht schon bei Dedekind") ([12], p. 68). In commenting on them, I will thus be considering their roots in Dedekind's work and in that of others from which she drew inspiration and on which she built.

Emmy Noether contributed to the following major areas of algebra: invariant theory (1907-1919), commutative algebra (1920-1929), noncommutative algebra and representation theory (1927-1933), and applications of noncommutative algebra to problems in commutative algebra (1932-1935). She thus dealt with just about the whole range of subject-matter of the algebraic tradition of the 19th and early 20th centuries (with the possible exception of group theory proper). What is significant is that she transformed that subject-matter, thereby originating a new algebraic tradition — what has come to be known as modern or abstract algebra.

I will now discuss Emmy Noether's contributions to each of the above areas.

INVARIANT THEORY

Emmy Noether's statement (quoted above), that her ideas are already in Dedekind's work, could, with equal validity, have been put as "It all started with Gauss". Indeed, invariant theory dates back to Gauss' study of binary quadratic forms in his *Disquisitiones Arithmeticae* of 1801. Gauss defined an equivalence relation on such forms and showed that the discriminant is an invariant of the form under equivalence (see [1]). A second important source of invariant theory is projective geometry, which originated in the 1820s. A significant problem was to distinguish euclidean from projective properties of geometric figures. The projective properties turned out to be those invariant under "projective transformations" (see [26], [31]).

Formally, invariant theory began with Cayley and Sylvester in the late 1840s. Cayley used it to bring to light the deeper connections between metric and projective geometry (see [10]). Although important connections with geometry were maintained throughout the 19th and early 20th centuries, invariant theory soon became an area of investigation independent of its relations to geometry. In fact, it became an important branch of *algebra* in the second half of the 19th century. To Sylvester "all algebraic inquiries, sooner or later, end at the Capitol of modern algebra over whose shining portal is inscribed the Theory of Invariants" ([26], p. 930).

An important problem of the abstract theory of invariants was to discover invariants of various "forms".[1]) Many of the major mathematicians of the second half of the 19th century worked on the computation of invariants of specific forms. This led to the major problem of invariant theory, namely to determine a complete system of invariants (a basis) for a given form; i.e., to find invariants of the form — it was conjectured that finitely many would do — such that every other invariant could be expressed as a combination of these. Cayley showed in 1856 that the finitely many invariants he had found earlier for binary quartic forms (i.e., forms of degree four in two variables) are a complete system. About ten years later Gordan proved that every binary form (of any degree) has a finite basis. Gordan's proof of this important result was computational – he *exhibited* a complete system of invariants.[2]) In 1888 Hilbert astonished the mathematical world by announcing a new, conceptual, approach to the problem of invariants. The idea was to consider, instead of invariants, expressions in a finite number of variables, in short, the polynomial ring in those variables. Hilbert then proved what came to be

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[1]) E.g., a *binary form* is an expression of the form $f(x_1, x_2) = a_0x_1^n + a_1x_1^{n-1}x_2 + \ldots + a_nx_2^n$. If this form is transformed by a linear transformation T of the variables x_1 and x_2 into the form $F(X_1, X_2) = A_0X_1^n + A_1X_1^{n-1}X_2 + \ldots + A_nX_2^n$, then any function I of the coefficients of f which satisfies the relation $I(A_0, \ldots, A_n) = r^kI(a_0, \ldots, a_n)$ is called an *invariant* of f under T (r denotes the determinant of T).

[2]) Weyl observed that "there exist papers of his [Gordan's] where twenty pages of formulas are not interrupted by a single word; it is told that in all his papers he himself wrote the formulas only, the text being added by his friends" ([41], p. 117).

known as the Basis Theorem, namely that every ideal in the ring of polynomials in finitely many variables has a finite basis. A corollary was that every form (of any degree, in any number of variables) has a finite complete system of invariants. Gordan's reaction to Hilbert's proof, which did not explicitly exhibit the complete system of invariants, was that "this is not mathematics; it is theology" ([26], p. 930).[1])

Emmy Noether's thesis, written under Gordan in 1907, was entitled "On Complete Systems of Invariants for Ternary Biquadratic Forms". The thesis was computational, in the style of Gordan's work. It ended with a table of the complete system of 331 invariants for such a form. Noether was later to describe her thesis as "a jungle of formulas" ([24], p. 11).[2])

Emmy Noether obtained, however, several notable results on invariants during the 1910s. First, using the methods she had developed in two papers (in 1915 and 1916) on the subject, she made a significant contribution to the problem, first posed by Dedekind, of finding a Galois extension of a given number field with a prescribed Galois group.[3]) Second, during her work in Göttingen on differential invariants, she used the calculus of variations to obtain the so-called Noether Theorem, still important in mathematical physics (see [7], p. 125). The physicist Fez Gursey says of this contribution ([22], p. 23):

> The key to the relation of symmetry laws to conservation laws in physics is Emmy Noether's celebrated theorem which states that a dynamical system described by an action under a Lie group with n parameters admits n invariants (conserved quantities) that remain constant in time during the evolution of the system.

Alexandrov summarizes her work on invariants by noting that it "would have been enough... to earn her the reputation of a first class mathematician" ([2], p. 156).

What was the route that led Emmy Noether from the computational theory of invariants to the abstract theory of rings and modules?[4]) In 1910 Gordan retired from the University of Erlangen and was soon replaced by Ernst

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[1]) Later Hilbert gave a constructive proof of his result which, however, he did not consider significant, but which elicited from Gordan the statement: "I have convinced myself that theology also has its advantages" ([26], p. 930).

[2]) When asked in 1932 to review a paper on invariants, she refused, declaring "I have completly forgotten all of the symbolic calculations I ever learned" ([12], p. 18).

[3]) The problem, in this generality, is still unresolved, although it has been solved for symmetric and solvable groups (see [7], p. 115).

[4]) "A greater contrast is hardly imaginable than between her first paper, the dissertation, and her works of maturity", remarks Weyl ([41], p. 120).

Fischer. He, too, was a specialist in invariant theory, but invariant theory of the Hilbert persuasion. Emmy Noether came under his influence and gradually made the change from Gordan's algorithmic approach to invariant theory to Hilbert's conceptual approach. Later work on invariants brought her in contact with the famous joint paper of Dedekind and Weber (see p. 115 below) on the arithmetic theory of algebraic functions. She became "sold" on Dedekind's approach and ideas, and this determined the direction of her future work.

COMMUTATIVE ALGEBRA

The two major sources of commutative algebra are algebraic geometry and algebraic number theory. Emmy Noether's two seminal papers of 1921 and 1927 on the subject can be traced, respectively, to these two sources. In these papers, entitled, respectively, Ideal Theory in Rings (Idealtheorie in Ringbereichen) and Abstract Development of Ideal Theory in Algebraic Number Fields and Function Fields (Abstrakter Aufbau der Idealtheorie in algebraischen Zahl-und Funktionenkörpern), she broke fundamentally new ground, originating "a new and epoch-making style of thinking in algebra" ([41], p. 130).

Algebraic geometry had its origins in the study, begun in the early 19th century, of abelian functions and their integrals. This analytic approach to the subject gradually gave way to geometric, algebraic, and arithmetic means of attack. In the algebraic context, the main object of study is the ring of polynomials $k[x_1, x_2, \ldots, x_n]$, k a field (in the 19th century k was the field of real or complex numbers). Hilbert in the 19th century, and Lasker and Macauley in the early 20th century, had shown that in such a ring every ideal is a finite intersection of primary ideals, with certain uniqueness properties.[1]) (Geometrically, the result says that every variety is a unique, finite, union of irreducible varieties.) In her 1921 paper Emmy Noether generalized this result to arbitrary commutative rings with the ascending chain condition (a.c.c.).[2]) Her main result was that in such a ring every ideal is a finite intersection (with accompanying uniqueness properties) of primary ideals. (See [14] for historical and [3] for technical details.)

What was so significant about this paper which (we recall) MacLane singled out as marking the beginning of abstract algebra as a conscious discipline?

[1]) An ideal I in a commutative ring R is called *primary* if $xy \in I$ implies $x \in I$ or $y^t \in I$ for some positive integer t. The concept of primary ideal is an extension to rings of prime power for the integers.

[2]) A commutative ring R satisfies the *ascending chain condition* if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ terminates; i.e., $I_n = I_{n+1} = \ldots$ for some positive integer n.

First and foremost was the isolation of the a.c.c. as the crucial concept needed in the proof of the main result. In fact, the proof "rested entirely on elementary consequences of the chain condition and... [was] startling in... simplicity" ([22], p. 13). Earlier proofs (of the corresponding result for polynomial rings) involved considerable computation, such as elimination theory and the geometry of algebraic sets.

The a.c.c. did not originate with Emmy Noether. Dedekind (in 1894) and Lasker (in 1905) used it, but in concrete settings of rings of algebraic integers and of polynomials, respectively. Moreover, the a.c.c. was for them incidental rather than of major consequence. Noether's isolation of the a.c.c. as an important concept was a watershed. Thanks to her work, rings with the a.c.c., now called noetherian rings [1]), have been singled out for special attention. In fact, commutative algebra has been described as the study of (commutative) noetherian rings. As such, the subject had its formal genesis in Emmy Noether's 1921 paper.

Another fundamental concept with Emmy Noether highlighted in the 1921 paper is that of a ring. This concept, too, did not originate with her. Dedekind (in 1871) introduced it as a subset of the complex numbers closed under addition, substraction, and multiplication, and called it an "order". Hilbert (in 1897), in his famous Report on Number Theory (Zahlbericht), coined the term "ring", but only in the context of rings of integers of algebraic number fields. Fraenkel (in 1914) gave essentially the modern definition of ring, but postulated two extraneous conditions. Noether (in the 1921 paper) gave the definition in current use (given also, apparently, by Sono in 1917, but this went unnoticed).

But it was not merely Noether's *definition* of the concept of ring which proved important. Through her groundbreaking papers in which the concept of ring played an essential role (and of which the 1921 paper was an important first), she brought this concept into prominence as a central notion of algebra. It immediately began to serve as the starting point for much of abstract algebra, taking its rightful place alongside the concepts of group and field, already reasonably well established at that time.

Noether also began to develop in the 1921 paper a general theory of ideals for commutative rings. Notions of prime, primary, and irreducible ideal, of intersection and product of ideals, of congruence modulo an ideal — in short, much of the machinery of ideal theory, appears here.

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[1]) A term coined in 1943 by Chevalley.

Toward the end of the paper she defined the concept of *module* over a noncommutative ring and showed that some of the earlier decomposition results for ideals carry over to submodules. (I will discuss modules in connection with Noether's work in noncommutative algebra.)

To summarize, the 1921 paper introduced and gave prominence to what came to be some of the basic concepts of abstract algebra, namely ring, module, ideal, and the a.c.c. Beyond that, it introduced, and began to show the efficacy of, a new way of doing algebra — abstract, axiomatic, conceptual. No mean accomplishment for a single paper! (See [19] and [22] for further details.)

Emmy Noether's 1927 paper had its roots in algebraic number theory and, to a lesser extent, in algebraic geometry. The sources of algebraic number theory are Gauss' theory of quadratic forms of 1801, his study of biquadratic reciprocity of 1832 (in which he introduced the Gaussian integers), and attempts in the early 19th century to prove Fermat's Last Theorem. In all cases the central issue turned out to be unique factorization in rings of integers of algebraic number fields.[1]) When examples of such rings were found in which unique factorization fails,[2]) the problem became to try to "restore", in some sense, the "paradise lost". This was achieved by Dedekind in 1871 (and, in a different way, by Kronecker in 1882) when he showed that unique factorization can be reestablished if one considers factorization of ideals (which he had introduced for this purpose) rather than of elements. His main result was that if R is the ring of integers of an algebraic number field, then every ideal of R is a unique product of prime ideals.[3]) (See [6] for historical and [34] for technical details.)

Riemann introduced "Riemann surfaces" in the 1850s in order to facilitate the study of (multivalued) algebraic functions. His methods were, however, nonrigorous, and depended on physical considerations. In 1882 Dedekind and Weber wrote an all-important paper whose aim was to give rigorous, algebraic, expression to some of Riemann's ideas on complex

[1]) An *algebraic number field* is a finite extension of the rationals, $Q(\alpha) = \{a_0 + a_1\alpha + \ldots + a_n\alpha^n\colon a_i \in Q, \alpha \text{ an algebraic number}\}$. The *ring of integers* of $Q(\alpha)$ consists of the elements of $Q(\alpha)$ which are roots of *monic* polynomials with *integer* coefficients. See [1] for details.

[2]) $R = \{a + b\sqrt{-5}\colon a, b \in \mathbf{Z}\}$ is such an example. Here

$$6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

are two distinct decomposition of 6 as a product of primes of R.

[3]) An ideal I of a ring R is said to be *prime* if $xy \in I$ implies $x \in I$ or $y \in I$. Prime ideals are generalizations of primes in the ring of integers.

function theory, in particular to his notion of a Riemann surface. Their idea was to establish an analogy between algebraic number fields and algebraic function fields, and to carry over the machinery and results of the former to the latter. They succeeded admirably, giving (among other things) a purely algebraic definition of a Riemann surface, and an algebraic proof of the fundamental Riemann-Roch Theorem. At least as importantly, they pointed to what proved to be a most fruitful idea, namely the interplay between algebraic number theory and algebraic geometry.

More specifically, just as in algebraic number theory one associates an algebraic number field $Q(\alpha)$ with a given algebraic number, so in algebraic geometry one associates an algebraic function field $\mathbf{C}(x, y)$ with a given algebraic function. $\mathbf{C}(x, y)$ consists of polynomials in x and y with complex coefficients, where y satisfies a polynomial equation with coefficients in $\mathbf{C}(x)$ (i.e., y is algebraic over $\mathbf{C}(x)$).[1] If A is the "ring of integers" of $\mathbf{C}(x, y)$ (i.e., A consists of the roots in $\mathbf{C}(x, y)$ of *monic* polynomials with coefficients in $\mathbf{C}[x]$), then a major result of the Dedekind-Weber paper is that every ideal in A is a unique product of prime ideals. (See [14] and [26] for historical, and [9] and [16] for technical, details.)

In her 1927 paper Emmy Noether generalized the above decomposition results for algebraic number fields and function fields to commutative rings. In fact, she characterized those commutative rings in which every ideal is a unique product of prime ideals. Such rings are now called *Dedekind domains*. She showed that R is a Dedekind domain if and only if (1) R statisfies the a.c.c., (2) R/I statisfies the d.c.c. for every nonzero ideal I of R, (3) R is an integral domain (i.e., it has an identity and no zero divisors), and (4) R is integrally closed in its field of quotients. Condition (4) proved particularly significant since it singled out the basic notion of integral dependence (related to that of integral closure).[2]) This concept (already present in Dedekind's work on algebraic numbers) has proved to be of fundamental importance in commutative algebra. As Gilmer notes, "the concept of integral dependence is to *Aufbau* [Noether's 1927 paper] what the a.c.c. is to *Idealtheorie* [her 1921 paper]" ([19], p. 136). Among other basic results she proved in this paper are: (a) the (by now standard) isomorphism and homomorphism theorems for rings and modules, (b) that a module M has a composition series if and only if it

[1]) $\mathbf{C}(x, y)$ is an extension field of $\mathbf{C}$ of transcendence degree 1; i.e., x is transcendental over $\mathbf{C}$ and y is algebraic over $\mathbf{C}(x)$. Thus, in analogy with the algebraic number field $Q(\alpha)$, $\mathbf{C}(x)$ corresponds to Q and y to α.

[2]) Let $R \subseteq S$ be rings. An element $s \in S$ is *integrally dependent* on R (or is integral over R) if it satisfies a monic polynomial with coefficients in R. R is *integrally closed* in S if every element of S which is integral over R belongs to R.

satisfies both the a.c.c. and d.c.c., (c) that if an R-module M is finitely generated and R satisfies the a.c.c. (d.c.c.), then so does M.

To summarize Emmy Noether's contributions to commutative algebra: in addition to proving important results, she introduced concepts and developed techniques which have become standard tools of the subject. In fact, her 1921 and 1927 papers, combined with those of Krull of the 1920s, are said to have created the subject of commutative algebra.

Noncommutative algebra and representation theory

Before her ideas in commutative algebra had been fully assimilated by her contemporaries, Emmy Noether turned her attention to the other major algebraic subjects of the 19th and early 20th centuries, namely hypercomplex number systems (what we now call associative algebras) and groups (in particular, group representations). She extended and unified these two subjects through her abstract, conceptual approach, in which module-theoretic ideas that she had used in the commutative case played a crucial role.

The theory of hypercomplex systems began with Hamilton's 1843 introduction of the quaternions. At the end of the 19th century, E. Cartan, Frobenius, and Molien gave structure theorems for such systems over the real and complex numbers, and in 1907 Wedderburn extended these to hypercomplex systems over arbitrary fields. In the spirit of Emmy Noether's work in commutative algebra, Artin extended Wedderburn's results to (noncommutative, semisimple) rings with the descending chain condition. (See [25] for details.)

Groups were the first algebraic systems to be developed extensively. By the end of the 19th century they began to be studied abstractly. An important tool in that study was representation theory, developed by Burnside, Frobenius, and Molien in the 1890s (see [20]). The idea was to study, instead of the abstract group, its concrete representations in terms of matrices (A *representation* of a group is a homomorphism of the group into the group of invertible matrices of some given order.)

In her 1929 paper Hypercomplex Numbers and Representation Theory (Hyperkomplexe Grössen und Darstellungstheorie) Emmy Noether framed group representation theory in terms of the structure theory of hypercomplex systems. The main tool in this approach was the *module.* The idea was to associate with each representation ϕ of G by invertible matrices with entries in some field k, a $k(G)$-module V called the *representation module* of ϕ($k(G)$ is the *group algebra* of G over k). Conversely, any $k(G)$-module M gives rise to a representation ψ of G.[1]) This establishes a one-one correspondence between representations of G (over k) and $k(G)$-modules. The standard concepts of representation theory can now be phrased in terms of modules. For example, two representations are equivalent if and only if their representation modules are isomorphic; a representation is irreducible if and only if its representation module is simple. The techniques of module theory, and the structure theory of hypercomplex systems (applied to the hypercomplex system $k(G)$) can now be used to "recast the foundations of group representation theory" ([27], p. 150). (See [27] for historical and [11] for technical details.)

Noether's work in this area created a very effective conceptual framework in which to study representation theory. For example, while the (computational) classical approach to representation theory is valid only over the field of complex numbers (or, at best, over an algebraically closed field of characteristic 0), Noether's approach remains meaningful for any field (of any characteristic). The use of general fields in representation theory became important in the 1930s when Brauer began his pioneering studies of *modular representations* (i.e., those in which the characteristic of the field divides the order of the group). Noether's ideas also "planted the seed of modern integral representation theory" ([27], p. 152), that is, representation theory over commutative rings rather than over fields. Noether herself extended the representation theory of groups to that of semi-simple artinian rings; here she needed the concept of a *bimodule.*

A word about modules, which were so central in Emmy Noether's work in both commutative and noncommutative algebra. Dedekind, in connection with his 1871 work in algebraic number theory, was the first to use the term "module", but to him it meant a subgroup of the additive group of complex numbers (i.e., a Z-module); in 1894 he developed an extensive theory of such modules. Lasker, in his 1905 work on decomposition of polynomial rings, used the terms "module" and "ideal" interchangeably (the former he applied to polynomial rings over **C**, the latter to such rings over **Z**). Noether was the first to use the notion of module abstractly (with a ring as domain of operators) and to recognize its potential. In fact, it is through her work that the concept of module became the central concept of algebra that it is today. Indeed, modules are important not only because of their unifying, but also because of their *linearizing,* power. (They are, after all, generalizations of vector

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[1]) In one direction, consider ϕ as a homomorphism of G into $L(V, V)$, the set of linear transformations of a vector space V over k. We turn V into a $k(G)$-module by defining $v \cdot g = \phi(g)(v)$, for $v \in V$, $g \in G$, and extending by linearity to all of $k(G)$. In the other direction, define $\psi: G \to L(M, M)$ by $\psi(g)(m) = m \cdot g$. See [11], Chapter II, for details.

spaces, and many of the standard vector-space constructions, such as subspace, quotient space, direct sum, and tensor product carry over to modules.)[1]) In fact, the importance of the invention of *homological algebra* was that it carried the process of linearization far forward by developing tools for its implementation. (E.g., the functors "Ext" and "Tor" measure the extent to which modules over general rings "misbehave" when compared to modules over fields, viz. vector spaces; see [8].)

APPLICATIONS OF NONCOMMUTATIVE TO COMMUTATIVE ALGEBRA

Noether believed that the theory of noncommutative algebras is governed by simpler laws than that of commutative algebra. In her 1932 plenary address at the International Congress of Mathematicians in Zurich, entitled Hypercomplex Systems and their Relations to Commutative Algebra and Number Theory (Hyperkomplexe Systeme in ihren Beziehungen zur kommutativen Algebra und Zahlentheorie), she outlined a program putting that belief into practice. Her program has been called "a foreshadowing of modern cohomology theory" ([35], p. 8). The ideas on factor sets contained therein were soon used by Hasse and Chevalley "to obtain some of the main results on global and local class field theory" ([22], p. 26). Noether's own immediate objective was to apply the theory of central simple algebras (as developed by her, Brauer, and others) to problems in class field theory. (See [7], [35], and [36].)

Some of her ideas (and those of others) on the interplay between commutative and noncommutative algebra had already recently born fruit with the proof of the celebrated Albert-Brauer-Hasse-Noether Theorem. This result, called by Jacobson "one of the high points of the theory of algebras" ([22], p. 21), gives a complete description of finite-dimensional division algebras over algebraic number fields.[2]) It is important in the study of finite-dimensional algebras and of group representations.

To bring out the context of the above theorem, it should be noted that Wedderburn's 1907 structure theorems for finite-dimensional algebras reduced their study to that of nilpotent algebras and division algebras. Since the unravelling of the structure of the former seemed (and still seems, despite considerable progress) "hopeless", attention focussed on the latter.

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[1]) We know the power of linearization in analysis. Modules can be said to provide analogous power in algebra.

[2]) They are intimately related to the "cyclic" algebras studied earlier by Dickson (see [21], Vol. II, p. 480).

Considerable progress on the structure of division algebras was made in the late 1920s and early 1930s. The Albert-Brauer-Hasse-Noether Theorem was a high point of these researches. It should be stressed, however, that even today much is still unknown about finite-dimensional division algebras.

C. Her legacy

The concepts Emmy Noether introduced, the results she obtained, and the mode of thinking she promoted, have become part of our mathematical culture. As Alexandrov put it ([2], p. 158):

> It was she who taught us to think in terms of simple and general algebraic concepts — homomorphic mappings, groups and rings with operators, ideals — and not in cumbersome algebraic computations; and [she] thereby opened up the path to finding algebraic principles in places where such principles had been obscured by some complicated special situation...

Moreover, as Weyl noted, "her significance for algebra cannot be read entirely from her own papers; she had great stimulating power and many of her suggestions took shape only in the works of her pupils or co-workers" ([41], pp. 129-130). Indeed, Weyl himself acknowledged his indebtedness to her in his work on groups and quantum mechanics. Among others who have *explicitly* mentioned her influence on their algebraic works are Artin, Deuring, Hasse, Jacobson, Krull, and Kurosh.

Another important vehicle for the spread of Emmy Noether's ideas was the now-classic treatise of van der Waerden entitled "Modern Algebra", first published in 1930. (It was based on lectures of Noether and Artin — see [39].) Its wealth of beautiful and powerful ideas, brilliantly presented by van der Waerden, has nurtured a generation of mathematicians. The book's immediate impact is poignantly described by Dieudonné and G. Birkhoff, respectively:

> I was working on my thesis at that time; it was 1930 and I was in Berlin. I still remember the day that van der Waerden came out on sale. My ignorance in algebra was such that nowadays I would be refused admittance to a university. I rushed to those volumes and was stupefied to see the new world which opened before me. At that time my knowledge of algebra went no further than *mathématiques spéciales,* determinants, and a little on the solvability of equations and unicursal curves. I had graduated from the École Normale and I did not know what an ideal was, and only just knew what a group was! This gives you an idea of what a young French mathematician knew in 1930 ([13], p. 137).

> Even in 1929, its concepts and methods [i.e., of "modern algebra"] were still considered to have marginal interest as compared with those of analysis in most universities, including Harvard. By exhibiting their mathematical and philosophical unity and by showing their power as developed by Emmy Noether and her other younger colleagues (most notably E. Artin, R. Brauer, and H. Hasse), van der Waerden made "modern algebra" suddenly seem central in mathematics. It is not too much to say that the freshness and enthusiasm of his exposition electrified the mathematical world — especially mathematicians under 30 like myself ([4], p. 771).

A number of mathematicians and historians of mathematics have spoken of the "algebraization of mathematics" in this century (see e.g. [32]). Witness the *terminological* penetration of algebra into such fields as algebraic geometry, algebraic topology, algebraic number theory, algebraic logic, topological algebra, Banach algebras, von Neumann algebras, Lie groups, and normed rings. Emmy Noether's influence is evident directly in several of these fields and indirectly in others. She, too, seemed to have acknowledged that, when she said in a letter to Hasse in 1931: "My methods are really methods of working and thinking; this is why they have crept in everywhere anonymously" ([12], p. 61). Alexandrov and Hopf confirm this in the preface to their book on topology: "Emmy Noether's general mathematical insights were not confined to her specialty — algebra — but affected anyone who came in touch with her" ([12], p. 61). In fact, they, too (and, more importantly, algebraic topology) were major beneficiaries of her insights. As Jacobson notes ([22], p. v):

> As is quite well known, it was Emmy Noether who persuaded Alexandrov and... Hopf to introduce group theory into combinatorial topology and to formulate the then existing simplicial homology theory in group theoretic terms in place of the more concrete setting of incidence matrices.

Algebraic geometry is another area which witnessed very extensive algebraization beginning in the late 1920s and early 1930s. The testimonies of Zariski and van der Waerden, respectively, two of its foremost practitioners who were deeply involved in this process of algebraization, are revealing:

> It was a pity that my Italian teachers never told me there was such a tremendous development of the algebra which is connected with algebraic geometry. I only discovered this much later, when I came to the United States ([33], pp. 36-37).

> When I came to Göttingen in 1924, a new world opened up before me. I learned from Emmy Noether that the tools by which my questions [in

algebraic geometry] could be handled had already been developed... ([34], p. 32).[1])

Emmy Noether was a visiting professor in Moscow in 1928-1929. Alexandrov described the impact she has had on Pontryagin's work in the theory of continuous groups (topological algebra):

> It is not hard to follow the influence of Emmy Noether on the developing mathematical talent of Pontryagin; the strong algebraic flavour in Pontryagin's work undoubtedly profited greatly from his association with Emmy Noether ([2], p. 175).

I will give the last word to Garrett Birkhoff who, in an article in 1976 describing the rise of abstract algebra from 1936 to 1950, said the following ([5], p. 81):

> If Emmy Noether could have been at the 1950 [International] Congress [of Mathematicians], she would have felt very proud. Her concept of algebra had become central in contemporary mathematics. And it has continued to inspire algebraists ever since.

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[1]) To put this statement in perspective, van der Waerden precedes it with the following comments: "In the beginning of our century, many people felt that the theory of invariants was a mighty tool in algebraic geometry... I soon discovered that the real difficulties of algebraic geometry cannot be overcome by calculating invariants and covariants" ([39, p. 32).

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