

5. COMMENTS ON THE EXAMPLES IN TABLES II

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **38 (1992)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

The first equation gives

$$P\bar{P} + Q\bar{Q} = 11 + 11(1 + U + \bar{U})(1 + V + \bar{V}).$$

Substituting in the second equation, we get

$$(*) \quad P + \bar{Q} + 5\bar{P}Q + 4P\bar{Q} = -44 + 55(1 + U + \bar{U})(1 + V + \bar{V}).$$

Since $U\bar{U} = 1$, $U^2 = \bar{U}$ and $V\bar{V} = 3 + V + \bar{V}$, $V^2 = V + 2\bar{V}$, the constant terms in $\bar{P}Q$ and $P\bar{Q}$ are equal to $\sum_{i=0}^2 p_i q_i + 3 \sum_{j=3}^8 p_j q_j = c$, say. Hence, equating constant terms in the above equation (*), we must have

$$p_0 + q_0 + 9c = 11.$$

The only solution to this equation with all p_i, q_i being 0 or 1, is $p_0 = q_0 = 1$, $p_i = q_i = 0$ for $i = 1, \dots, 8$. This means $P = Q = 1$, contradicting (*).

5. COMMENTS ON THE EXAMPLES IN TABLES II

Difference sets with parameters $(v, k, \lambda) = (4n - 1, 2n - 1, n - 1)$ are usually called *Hadamard difference sets*. Our purpose here is to discuss the classification of these cyclic difference sets for $2 \leq n \leq 100$.

In many cases where $v = 4n - 1$ is a prime p , the quadratic residue difference set, which we denote by $QR(p)$ is unique for the given values of the parameters. This is obviously the case if the multiplier m has order

$k = \frac{1}{2}(v - 1)$ in $(\mathbf{Z}/v\mathbf{Z})^*$. Indeed, in this case, there are exactly 3 orbits of

multiplication by m in $\mathbf{Z}/v\mathbf{Z}$, namely $1 = \{0\}$, $M = \{1, m, m^2, \dots, m^{k-1}\}$ and $\bar{M} = \{-1, -m, \dots, -m^{k-1}\}$. Thus the only choice for D is $D = M$ or $D = \bar{M}$, which are isomorphic under conjugation $\sigma: \mathbf{Z}/v\mathbf{Z} \rightarrow \mathbf{Z}/v\mathbf{Z}$, $\sigma(a) = -a$.

In our Table II, this situation happens for $n = 3, 5, 6, 12, 15, 17, 18, 20, 21, 27, 33, 35, 41, 42, 45, 48, 53, 57, 60, 63, 66, 68, 77, 87, 90$ and 96 .

The remaining cases where $v = 4n - 1$ is a prime p (for $2 \leq n \leq 100$) have been shown to lead to a single difference set, namely $QR(p)$, by machine enumeration of the various choices of D as a union of orbits under multiplication by a multiplier m . This includes the cases $n = 26$ (multiplier 8), $n = 38$ (multiplier 19), $n = 50$ (multiplier 5), $n = 78$ (multiplier 13), $n = 83$ (multiplier 83), and $n = 95$ (multiplier 5). By far, the most difficult case (for the machine) occurs with $n = 38$, which required the examination of 37 442 160 combinations of multiplier orbits.

The case $n = 36$, also leads by machine enumeration to the single difference set $TP(11, 13)$ of twin-prime type with parameters $(143 = 11 \cdot 13, 71, 35)$.

The only other values of $n (\leq 100)$ for which $v = 4n - 1$ is *not* a prime and a Hadamard cyclic difference set with parameters $(4n - 1, 2n - 1, n - 1)$ does exist are powers of 2. The examples for $n = 2, 4$ and 8 are easily seen to be unique.

For $n = 16$, there are 2 isomorphism classes of Hadamard difference sets with parameters $(63, 31, 15)$: Both have multiplier $m = 2$, and denoting by X_a the orbit of a under multiplication by 2 in $\mathbf{Z}/63\mathbf{Z}$, they are

$$D_0 = 1 + X_{-25} + X_{-9} + X_1 + X_3 + X_7 + X_9 ,$$

which is isomorphic to $\mathbf{P}^5(\mathbf{F}_2)$, and

$$D_1 = 1 + X_{-9} + X_{-1} + X_1 + X_3 + X_9 + X_{25} ,$$

which is of type *GMW*.

The difference sets D_0 and D_1 are not isomorphic, even as block designs, as can be seen by computing the cardinalities of the intersection of triples of blocks of D_1 , giving the enumerating polynomial

$$10584t^6 + 19656t^7 + 3528t^8 + 5880t^9 + 63t^{15} ,$$

in contrast to $39060t^7 + 651t^{15}$ for $\mathbf{P}^5(\mathbf{F}_2)$. (The coefficient of t^i being the multiplicity of triple intersections of cardinality i .)

For $n = 32$, L. Baumert and H. Fredricksen have found that there are exactly 6 non-isomorphic examples. (See [BF].) Three of these, $QR(127)$, $\mathbf{P}^6(\mathbf{F}_2)$ and $MH(127)$ are members of the classical families.

For $n = 64$, we have found that there exist exactly 4 examples (up to isomorphism). One of them is $\mathbf{P}^7(\mathbf{F}_2)$, another one is of type *GMW*. The other two seem to be new.

All 4 of them have multiplier 2, which is of order 8 modulo $v = 255$. They all contain the union U of the multiplier orbits of length < 8 , viz.

$$U = \{0\} + \{85, -85\} + \{51, 102, -51, -102\} + \{17, 34, 68, -119\} \\ + \{-17, -34, -68, 119\} .$$

Denoting by (a_1, \dots, a_{14}) the union $\sum_{i=1}^{14} X_{a_i}$ of the orbits

$$X_a = \{a, 2a, \dots, 2^7a\} ,$$

the 4 examples are of the form $D_i = U + V_i$, where

$$V_0 = (-19, -9, -7, -1, 1, 3, 7, 13, 19, 23, 25, 27, 37, 45),$$

$$V_1 = (-43, -27, -25, -13, -9, -5, -3, 7, 11, 13, 19, 23, 27, 43),$$

$$V_2 = (-43, -27, -23, -13, -11, -3, 1, 3, 7, 13, 15, 25, 37, 43),$$

$$V_3 = (-43, -23, -21, -11, -7, -3, 7, 9, 11, 15, 19, 25, 37, 43).$$

The difference sets D_2 and D_3 appear to be exotic. D_0 is isomorphic to $\mathbf{P}^7(\mathbf{F}_2)$. Finally, D_1 is of type *GMW*, and can be constructed as follows.

Let $L = \mathbf{F}_{256}$ be the extension of degree 8 over $F = \mathbf{F}_2$. We will use the trace $Tr = Tr_{L/F}: L \rightarrow F$ given by $Tr(\gamma) = \sum_{i=0}^7 \gamma^{2^i}$. The extension L/F is defined by the irreducible polynomial $x^8 + x^4 + x^3 + x^2 + 1 \in \mathbf{F}[x]$. The multiplicative group \mathbf{F}_{256}^* is generated by any root α of this polynomial. The Hall polynomial $D_0(x)$ of D_0 is then given by

$$D_0(x) = \sum_{i=0}^{254} d_i x^i \in \mathbf{Z}[x]/(x^{255} - 1),$$

where

$$d_i = \begin{cases} 0 & \text{if } Tr(\alpha^i) \neq 0 \\ 1 & \text{if } Tr(\alpha^i) = 0. \end{cases}$$

Thus a block of the difference set is the hyperplane $\ker(Tr) \subset \mathbf{F}_{256} = \mathbf{F}_2^8$.

Under the identification

$$\mathbf{Z}/255\mathbf{Z} \rightarrow \mathbf{F}_{256}^*$$

given by $i \mapsto \alpha^i$, the multiplication by 2 in $\mathbf{Z}/255\mathbf{Z}$ becomes the Frobenius automorphism in the extension $\mathbf{F}_{256}/\mathbf{F}_2$. The block $\ker(Tr)$ is a union of orbits under the action of the multiplier.

In order to construct D_1 , the example of type *GMW*, we need the intermediate extension $K = \mathbf{F}_{16}$, $F \subset K \subset L$. Set $\beta = \alpha^{17}$, a generator of $K^* = \mathbf{F}_{16}^*$. Denote by $tr = tr_{K/F}: K \rightarrow F$ the trace.

Consider the complementary polynomial $D'_0(x) = T - D_0(x)$, where $T = \sum_{i=0}^{254} x^i \in \mathbf{Z}[x]/(x^{255} - 1)$. The crucial point is to observe that $D'_0(x)$ splits as

$$D'_0(x) = \Omega(x) \cdot \theta_0(x^{17}) \in \mathbf{Z}[x]/(x^{255} - 1),$$

where $\theta_0(y) = \sum_{j=0}^{14} a_j y^j$ with

$$a_j = \begin{cases} 0 & \text{if } tr(\beta^j) = 0 \\ 1 & \text{if } tr(\beta^j) \neq 0, \end{cases}$$

and $\Omega(x) = x^7 + x^{14} + \cdots + x^{246}$. Here,

$$\theta_0(y) = y + y^2 + y^3 + y^4 + y^6 + y^8 + y^9 + y^{12}.$$

Now define

$$D'_1(x) = \Omega(x) \cdot \theta_1(x^{17}),$$

where $\theta_1(y) = \theta_0(y^{-1})$. Then $D_1(x) = T - D'_1(x)$ is the Hall polynomial of the difference set D_1 .

The fact that D_0 , D_1 , D_2 and D_3 are not isomorphic, even as block designs, can again be seen by determining the cardinalities of all triple intersections of blocks, for each D_i . Denoting by P_i the corresponding enumerating polynomial of triple intersections for D_i , we have

$$P_0 = 2720340t^{31} + 10795t^{63}$$

$$P_1 = 979200t^{29} + 823140t^{31} + 734400t^{33} + 183600t^{35} \\ + 10200t^{39} + 595t^{63}$$

$$P_2 = 9180t^{25} + 8160t^{26} + 45900t^{27} + 163200t^{28} + 342720t^{29} \\ + 514080t^{30} + 518160t^{31} + 465120t^{32} + 358020t^{33} \\ + 179520t^{34} + 81090t^{35} + 18360t^{36} + 18360t^{37} + 6120t^{38} \\ + 3145t^{39}$$

$$P_3 = 4080t^{25} + 14280t^{26} + 40800t^{27} + 142800t^{28} + 385560t^{29} \\ + 403920t^{30} + 692580t^{31} + 424320t^{32} + 352920t^{33} + 128520t^{34} \\ + 79050t^{35} + 32640t^{36} + 9180t^{37} + 12240t^{38} + 7225t^{39} + 1020t^{45}.$$

TABLE I

Case $\gamma = +1$:

*Non-existence of a cyclic difference set
with parameters $(2t(t+1) + 1, t^2, \frac{1}{2}t(t-1))$ for $3 \leq t \leq 100$.
(The case $t = 50$ is still undecided.)*

t	(v, k, λ)	$n = k - \lambda$	reason for non-existence
3	$(5^2, 9, 3)$	$2 \cdot 3$	$2^2 \equiv -1 \pmod{5}$
4	$(41, 16, 6)$	$2 \cdot 5$	$5^{10} \equiv -1 \pmod{41}$
5	$(61, 25, 10)$	$3 \cdot 5$	$3^5 \equiv -1 \pmod{61}$
6	$(5 \cdot 17, 36, 15)$	$3 \cdot 7$	$3^2 \equiv -1 \pmod{5}$
7	$(113, 49, 21)$	$2^2 \cdot 7$	$7^7 \equiv -1 \pmod{113}$
8	$(5 \cdot 29, 64, 28)$	$2^2 \cdot 3^2$	$2^{14} \equiv -1 \pmod{145}$
9	$(181, 81, 36)$	$3^2 \cdot 5$	$5 \equiv 3^6 \pmod{181}$ would be multiplier
10	$(13 \cdot 17, 100, 45)$	$5 \cdot 11$	$5^2 \equiv -1 \pmod{13}$
11	$(5 \cdot 53, 121, 55)$	$2 \cdot 3 \cdot 11$	$2^2 \equiv -1 \pmod{5}$
12	$(3 \cdot 13, 144, 66)$	$2 \cdot 3 \cdot 13$	$2^{78} \equiv -1 \pmod{313}$
13	$(5 \cdot 73, 169, 78)$	$7 \cdot 13$	$7^2 \equiv -1 \pmod{5}$
14	$(421, 196, 91)$	$3 \cdot 5 \cdot 7$	$5^{105} \equiv -1 \pmod{421}$

TABLE I (continued)

t	(v, k, λ)	$n = k - \lambda$	reason for non-existence
15	(13 · 37, 225, 105)	$2^3 \cdot 3 \cdot 5$	$5^2 \equiv -1 \pmod{13}$
16	(5 · 109, 256, 120)	$2^3 \cdot 17$	$17^2 \equiv -1 \pmod{5}$
17	(613, 289, 136)	$3^2 \cdot 17$	$17^{51} \equiv -1 \pmod{613}$
18	(5 · 137, 324, 153)	$3^2 \cdot 19$	$19 \equiv -1 \pmod{5}$
19	(761, 361, 171)	$2 \cdot 5 \cdot 19$	$2^{190} \equiv -1 \pmod{761}$
20	(29 ² , 400, 190)	$2 \cdot 3 \cdot 5 \cdot 7$	$2^{14} \equiv -1 \pmod{29}$
21	(5 ² · 37, 441, 210)	$3 \cdot 7 \cdot 11$	$3^2 \equiv -1 \pmod{5}$
22	(1013, 484, 231)	$11 \cdot 23$	$11^{23} \equiv -1 \pmod{1013}$
23	(5 · 13 · 17, 529, 253)	$2^2 \cdot 3 \cdot 23$	$3^2 \equiv -1 \pmod{5}$
24	(1201, 576, 276)	$2^2 \cdot 3 \cdot 5^2$	$3^{150} \equiv -1 \pmod{1201}$
25	(1301, 625, 300)	$5^2 \cdot 13$	$5^{325} \equiv -1 \pmod{1301}$
26	(5 · 281, 676, 325)	$3^3 \cdot 13$	$13^2 \equiv -1 \pmod{5}$
27	(17 · 89, 729, 351)	$2 \cdot 3^3 \cdot 7$	$2^4 \equiv -1 \pmod{17}$
28	(5 ³ · 13, 784, 378)	$2 \cdot 7 \cdot 9$	$2^2 \equiv -1 \pmod{5}$
29	(1741, 841, 406)	$3 \cdot 5 \cdot 29$	$3^{435} \equiv -1 \pmod{1741}$
30	(1861, 900, 435)	$3 \cdot 5 \cdot 31$	$3^{155} \equiv -1 \pmod{1861}$
31	(5 · 397, 961, 465)	$2^4 \cdot 31$	$2^{22} \equiv -1 \pmod{1985}$
32	(2113, 1024, 496)	$2^4 \cdot 3 \cdot 11$	$3^{528} \equiv -1 \pmod{2113}$
33	(5 · 449, 1089, 528)	$3 \cdot 11 \cdot 17$	$3^2 \equiv -1 \pmod{5}$
34	(2381, 1156, 561)	$5 \cdot 7 \cdot 17$	$5^{119} \equiv -1 \pmod{2381}$
35	(2521, 1225, 595)	$2 \cdot 3^2 \cdot 5 \cdot 7$	$2^{630} \equiv -1 \pmod{2521}$
36	(5 · 13 · 41, 1296, 630)	$2 \cdot 3^2 \cdot 37$	$2^2 \equiv -1 \pmod{5}$
37	(29 · 97, 1369, 666)	$19 \cdot 37$	$19^{14} \equiv -1 \pmod{29}$
38	(5 · 593, 1444, 703)	$3 \cdot 13 \cdot 19$	$3^2 \equiv -1 \pmod{5}$
39	(3121, 1521, 741)	$2^2 \cdot 3 \cdot 5 \cdot 13$	$2^{78} \equiv -1 \pmod{3121}$
40	(17 · 193, 1600, 780)	$2^2 \cdot 5 \cdot 41$	$5^8 \equiv -1 \pmod{17}$
41	(5 · 13 · 53, 1681, 820)	$3 \cdot 7 \cdot 41$	$3^2 \equiv -1 \pmod{5}$
42	(3613, 1764, 861)	$3 \cdot 7 \cdot 43$	$3^{903} \equiv -1 \pmod{3613}$
43	(5 · 757, 1849, 903)	$2 \cdot 11 \cdot 43$	$2^2 \equiv -1 \pmod{5}$
44	(17 · 233, 1936, 946)	$2 \cdot 3^2 \cdot 5 \cdot 11$	$2^4 \equiv -1 \pmod{17}$
45	(41 · 101, 2025, 990)	$3^2 \cdot 5 \cdot 23$	$5^{10} \equiv -1 \pmod{41}$
46	(5 ² · 173, 2116, 1035)	$23 \cdot 47$	$23^2 \equiv -1 \pmod{5}$
47	(4513, 2209, 1081)	$2^3 \cdot 3 \cdot 47$	$3^{188} \equiv -1 \pmod{4513}$
48	(5 · 941, 2304, 1128)	$2^3 \cdot 3 \cdot 7^2$	$3^2 \equiv -1 \pmod{5}$
49	(13 ² · 29, 2401, 1176)	$5^2 \cdot 7^2$	$5^2 \equiv 7^6 \pmod{4901}$ would be multipli
50	(5101, 2500, 1225)	$3 \cdot 5^2 \cdot 17$	existence unsettled
51	(5 · 1061, 2601, 1275)	$2 \cdot 3 \cdot 13 \cdot 17$	$2^2 \equiv -1 \pmod{5}$
52	(37 · 149, 2704, 1326)	$2 \cdot 13 \cdot 53$	$2^{18} \equiv -1 \pmod{37}$
53	(5 ² · 229, 2809, 1378)	$3^3 \cdot 53$	$53^2 \equiv -1 \pmod{5}$
54	(13 · 457, 2916, 1431)	$3^3 \cdot 5 \cdot 11$	$5^2 \equiv -1 \pmod{13}$
55	(61 · 101, 3025, 1485)	$2^2 \cdot 5 \cdot 7 \cdot 11$	$5^{15} \equiv -1 \pmod{61}$
56	(5 · 1277, 3136, 1540)	$2^2 \cdot 3 \cdot 7 \cdot 19$	$3^2 \equiv -1 \pmod{5}$
57	(17 · 389, 3249, 1596)	$3 \cdot 19 \cdot 29$	$3^8 \equiv -1 \pmod{17}$

TABLE I (continued)

t	(v, k, λ)	$n = k - \lambda$	reason for non-existence
58	$(5 \cdot 37^2, 3364, 1653)$	$29 \cdot 59$	$29 \equiv -1 \pmod{5}$
59	$(73 \cdot 97, 3481, 1711)$	$2 \cdot 3 \cdot 5 \cdot 59$	$3^6 \equiv -1 \pmod{73}$
60	$(7321, 3600, 1770)$	$2 \cdot 3 \cdot 5 \cdot 61$	$2^{610} \equiv -1 \pmod{7321}$
61	$(5 \cdot 17, 3721, 1830)$	$31 \cdot 61$	$31^8 \equiv -1 \pmod{17}$
62	$(13 \cdot 601, 3844, 1891)$	$3^2 \cdot 7 \cdot 31$	$7^6 \equiv -1 \pmod{13}$
63	$(5 \cdot 1613, 3969, 1953)$	$2^5 \cdot 3^2 \cdot 7$	$7^2 \equiv -1 \pmod{5}$
64	$(53 \cdot 157, 4096, 2016)$	$2^5 \cdot 5 \cdot 13$	$5^{26} \equiv -1 \pmod{53}$
65	$(8581, 4225, 2080)$	$3 \cdot 5 \cdot 11 \cdot 13$	$3^{715} \equiv -1 \pmod{8581}$
66	$(5 \cdot 29 \cdot 61, 4356, 2145)$	$3 \cdot 11 \cdot 67$	$3^2 \equiv -1 \pmod{5}$
67	$(13 \cdot 701, 4489, 2211)$	$2 \cdot 17 \cdot 67$	$2^6 \equiv -1 \pmod{13}$
68	$(5 \cdot 1877, 4624, 2278)$	$2 \cdot 3 \cdot 17 \cdot 23$	$2^2 \equiv -1 \pmod{5}$
69	$(9661, 4761, 2346)$	$3 \cdot 5 \cdot 7 \cdot 23$	$7^{2415} \equiv -1 \pmod{9661}$
70	$(9941, 4900, 2415)$	$5 \cdot 7 \cdot 71$	$7^{2485} \equiv -1 \pmod{9941}$
71	$(5^2 \cdot 409, 5041, 2485)$	$2^2 \cdot 3^2 \cdot 71$	$2^{510} \equiv -1 \pmod{10225}$
72	$(10513, 5184, 2556)$	$2^2 \cdot 3^2 \cdot 73$	$2^{1314} \equiv -1 \pmod{10513}$
73	$(5 \cdot 2161, 5329, 2628)$	$37 \cdot 73$	$37^2 \equiv -1 \pmod{5}$
74	$(17 \cdot 653, 5476, 2701)$	$3 \cdot 5^2 \cdot 37$	$3^8 \equiv -1 \pmod{17}$
75	$(13 \cdot 877, 5625, 2775)$	$2 \cdot 3 \cdot 5^2 \cdot 19$	$2^6 \equiv -1 \pmod{13}$
76	$(5 \cdot 2341, 5776, 2850)$	$2 \cdot 7 \cdot 11 \cdot 19$	$2^2 \equiv -1 \pmod{5}$
77	$(41 \cdot 293, 5929, 2926)$	$3 \cdot 7 \cdot 11 \cdot 13$	$3^4 \equiv -1 \pmod{41}$
78	$(5^2 \cdot 17 \cdot 29, 6084, 3003)$	$3 \cdot 13 \cdot 79$	$3^2 \equiv -1 \pmod{5}$
79	$(12641, 6241, 3081)$	$2^3 \cdot 5 \cdot 79$	$5^{1580} \equiv -1 \pmod{12641}$
80	$(13 \cdot 997, 6400, 3160)$	$2^3 \cdot 3^4 \cdot 5$	$5^2 \equiv -1 \pmod{13}$
81	$(5 \cdot 2657, 6561, 3240)$	$3^4 \cdot 41$	$41^{332} \equiv -1 \pmod{2657}$
82	$(13613, 6724, 3321)$	$41 \cdot 83$	$41 \equiv 83^3 \pmod{13613}$ would be multiple
83	$(5 \cdot 2789, 6889, 3403)$	$2 \cdot 3 \cdot 7 \cdot 83$	$2^2 \equiv -1 \pmod{5}$
84	$(14281, 7056, 3486)$	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 17$	$2^{1190} \equiv -1 \pmod{14281}$
85	$(14621, 7225, 3570)$	$5 \cdot 17 \cdot 43$	$5^{3655} \equiv -1 \pmod{14621}$
86	$(5 \cdot 41 \cdot 73, 7396, 3655)$	$3 \cdot 29 \cdot 43$	$3^2 \equiv -1 \pmod{5}$
87	$(15313, 7569, 3741)$	$2^2 \cdot 3 \cdot 11 \cdot 29$	$3^{1276} \equiv -1 \pmod{15313}$
88	$(5 \cdot 13 \cdot 241, 7744, 3828)$	$2^2 \cdot 11 \cdot 89$	$89 \equiv -1 \pmod{5}$
89	$(37 \cdot 433, 7921, 3916)$	$3^2 \cdot 5 \cdot 89$	$5^{18} \equiv -1 \pmod{37}$
90	$(16381, 8100, 4005)$	$3^2 \cdot 5 \cdot 7 \cdot 13$	$13^{65} \equiv -1 \pmod{16381}$
91	$(5 \cdot 17 \cdot 197, 8281, 4095)$	$2 \cdot 7 \cdot 13 \cdot 23$	$2^2 \equiv -1 \pmod{5}$
92	$(109 \cdot 157, 8464, 4186)$	$2 \cdot 3 \cdot 23 \cdot 31$	$2^{18} \equiv -1 \pmod{109}$
93	$(5 \cdot 13 \cdot 269, 8649, 4278)$	$3 \cdot 31 \cdot 47$	$3^2 \equiv -1 \pmod{5}$
94	$(53 \cdot 337, 8836, 4371)$	$5 \cdot 19 \cdot 47$	$5^{26} \equiv -1 \pmod{53}$
95	$(17 \cdot 29 \cdot 37, 9025, 4465)$	$2^4 \cdot 3 \cdot 5 \cdot 19$	$3^8 \equiv -1 \pmod{17}$
96	$(5^3 \cdot 149, 9216, 4560)$	$2^4 \cdot 3 \cdot 97$	$3^2 \equiv -1 \pmod{5}$
97	$(19013, 9409, 4656)$	$7^2 \cdot 97$	$7^{4753} \equiv -1 \pmod{19013}$
98	$(5 \cdot 3881, 9604, 4753)$	$3^2 \cdot 7^2 \cdot 11$	$11^{97} \equiv -1 \pmod{3881}$
99	$(19801, 9801, 4851)$	$2 \cdot 3^2 \cdot 5^2 \cdot 11$	$2^{4950} \equiv -1 \pmod{19801}$
100	$(20201, 10000, 4950)$	$2 \cdot 5^2 \cdot 101$	$2^{5050} \equiv -1 \pmod{20201}$

TABLE II

Case $\gamma = -1$:Cyclic difference sets with parameters $(4n - 1, 2n - 1, n - 1)$, $2 \leq n \leq 100$

n	$(4n - 1, 2n - 1, n - 1)$	exists?	examples or comment to non-existence
2	(7, 3, 1)	Yes	$\mathbf{P}^2(\mathbf{F}_2) = QR(7)$
3	(11, 5, 2)	Yes	$QR(11)$
4	(3 · 5, 7, 3)	Yes	$TP(3, 5) = \mathbf{P}^3(\mathbf{F}_2)$
5	(19, 9, 4)	Yes	$QR(19)$
6	(23, 11, 5)	Yes	$QR(23)$
7	(3 ³ , 13, 6)	No	7 would be multiplier
8	(31, 15, 7)	Yes	$\mathbf{P}^4(\mathbf{F}_2)$ and $QR(31)$
9	(5 · 7, 17, 8)	Yes	$TP(5, 7)$
10	(3 · 13, 19, 9)	No	$2 \equiv -1 \pmod{3}$
11	(43, 21, 10)	Yes	$QR(43)$
12	(47, 23, 11)	Yes	$QR(47)$
13	(3 · 17, 25, 12)	No	$13^2 \equiv -1 \pmod{17}$
14	(5 · 11, 27, 13)	No	$2^2 \equiv -1 \pmod{5}$
15	(59, 29, 14)	Yes	$QR(59)$
16	(3 ² · 7, 31, 15)	Yes	$\mathbf{P}^5(\mathbf{F}_2)$ and GMW
17	(67, 33, 16)	Yes	$QR(67)$
18	(71, 35, 17)	Yes	$QR(71)$
19	(3 · 5 ² , 35, 18)	No	$19 \equiv -1 \pmod{5}$
20	(79, 39, 19)	Yes	$QR(79)$
21	(83, 41, 20)	Yes	$QR(83)$
22	(3 · 29, 43, 21)	No	$2 \equiv -1 \pmod{3}$
23	(7 · 13, 45, 22)	No	$23^3 \equiv -1 \pmod{13}$
24	(5 · 19, 47, 23)	No	$3^2 \equiv -1 \pmod{5}$
25	(3 ² · 11, 49, 24)	No	25 would be multiplier
26	(103, 51, 25)	Yes	$QR(103)$
27	(107, 53, 26)	Yes	$QR(107)$
28	(3 · 37, 55, 27)	No	$m = 7 \equiv 2^{32}$ would be multiplier
29	(5 · 23, 57, 28)	No	$29 \equiv -1 \pmod{5}$
30	(7 · 17, 59, 29)	No	$2^4 \equiv -1 \pmod{17}$
31	(3 · 41, 61, 30)	No	$31^5 \equiv -1 \pmod{41}$
32	(127, 63, 31)	Yes	$\mathbf{P}^6(\mathbf{F}_2)$, $QR(127)$, $MH(127)$, and 3 others
33	(131, 65, 32)	Yes	$QR(131)$
34	(3 ³ · 5, 67, 33)	No	$17 \equiv -1 \pmod{3}$
35	(139, 69, 34)	Yes	$QR(139)$
36	(11 · 13, 71, 35)	Yes	$TP(11, 13)$
37	(3 · 7 ² , 73, 36)	No	37 would be multiplier
38	(151, 75, 37)	Yes	$QR(151)$
39	(5 · 31, 77, 38)	No	$3^2 \equiv -1 \pmod{5}$
40	(3 · 53, 79, 39)	No	$5 \equiv -1 \pmod{3}$

TABLE 2 (continued)

n	$(4n-1, 2n-1, n-1)$	exists?	examples or comment to non-existence
41	(163, 81, 40)	Yes	$QR(163)$
42	(167, 83, 41)	Yes	$QR(167)$
43	$(3^2 \cdot 19, 85, 42)$	No	43 would be multiplier
44	$(5^2 \cdot 7, 87, 43)$	No	$m = 11 \equiv 2^{56} \pmod{175}$ would be multiplier
45	(179, 89, 44)	Yes	$QR(179)$
46	$(3 \cdot 61, 91, 45)$	No	$23 \equiv -1 \pmod{3}$
47	$(11 \cdot 17, 93, 46)$	No	$47^2 \equiv -1 \pmod{17}$
48	(191, 95, 47)	Yes	$QR(191)$
49	$(3 \cdot 5 \cdot 13, 97, 48)$	No	7 would be multiplier
50	(199, 99, 49)	Yes	$QR(199)$
51	$(7 \cdot 29, 101, 50)$	No	$3^3 \equiv -1 \pmod{7}$
52	$(3^2 \cdot 23, 103, 51)$	No	13 would be multiplier
53	(211, 105, 52)	Yes	$QR(211)$
54	$(5 \cdot 43, 107, 53)$	No	$2^2 \equiv -1 \pmod{5}$
55	$(3 \cdot 73, 109, 54)$	No	$5 \equiv -1 \pmod{3}$
56	(223, 111, 55)	Yes	$QR(223)$ and $MH(223)$
57	(227, 113, 56)	Yes	$QR(227)$
58	$(3 \cdot 7 \cdot 11, 115, 57)$	No	$2 \equiv -1 \pmod{3}$
59	$(5 \cdot 47, 117, 58)$	No	$59 \equiv -1 \pmod{5}$
60	(239, 119, 59)	Yes	$QR(239)$
61	$(3^5, 121, 60)$	No	61 would be multiplier
62	$(13 \cdot 19, 123, 61)$	No	$2^6 \equiv -1 \pmod{13}$
63	(251, 125, 62)	Yes	$QR(251)$
64	$(3 \cdot 5 \cdot 17, 127, 63)$	Yes	$\mathbf{P}^7(\mathbf{F}_2)$, GMW and 2 new ones
65	$(7 \cdot 37, 129, 64)$	No	$5^3 \equiv -1 \pmod{7}$
66	(263, 131, 65)	Yes	$QR(263)$
67	$(3 \cdot 89, 133, 66)$	No	67 would be multiplier
68	(271, 135, 67)	Yes	$QR(271)$
69	$(5^2 \cdot 11, 137, 68)$	No	$3^2 \equiv -1 \pmod{5}$
70	$(3^2 \cdot 31, 139, 69)$	No	$2 \equiv -1 \pmod{3}$
71	(283, 141, 70)	Yes	$QR(283)$ and $MH(283)$
72	$(7 \cdot 41, 143, 71)$	No	$m = 9 \equiv 2^{55} \equiv 3^2 \pmod{287}$ would be multiplier
73	$(3 \cdot 97, 145, 72)$	No	$73^{12} \equiv -1 \pmod{97}$
74	$(5 \cdot 59, 147, 73)$	No	$2^2 \equiv -1 \pmod{5}$
75	$(13 \cdot 23, 149, 74)$	No	$m = 3^3 \equiv 5^4 \pmod{299}$ would be multiplier
76	$(3 \cdot 101, 151, 75)$	No	$m = 19 \equiv 2^9 \pmod{303}$ would be multiplier
77	(307, 153, 76)	Yes	$QR(307)$
78	(311, 155, 77)	Yes	$QR(311)$
79	$(3^2 \cdot 5 \cdot 7, 157, 78)$	No	$79 \equiv -1 \pmod{5}$
80	$(11 \cdot 29, 159, 79)$	No	$5^7 \equiv -1 \pmod{29}$
81	$(17 \cdot 19, 161, 80)$	Yes	$TP(17, 19)$
82	$(3 \cdot 109, 163, 81)$	No	$2 \equiv -1 \pmod{3}$

TABLE II (continued)

n	$(4n-1, 2n-1, n-1)$	exists?	examples or comment to non-existence
83	(331, 165, 82)	Yes	QR(331)
84	$(5 \cdot 67, 167, 83)$	No	$3^2 \equiv -1 \pmod{5}$
85	$(3 \cdot 113, 169, 84)$	No	$5 \equiv -1 \pmod{3}$
86	$(7^3, 171, 85)$	No	$m = 43 \equiv 2^{144} \pmod{343}$ would be multiplier
87	(347, 173, 86)	Yes	QR(347)
88	$(3^3 \cdot 13, 175, 87)$	No	$11 \equiv -1 \pmod{3}$
89	$(5 \cdot 71, 177, 88)$	No	$89 \equiv -1 \pmod{5}$
90	(359, 179, 89)	Yes	QR(359)
91	$(3 \cdot 11^2, 181, 90)$	No	$7^5 \equiv -1 \pmod{11}$
92	(367, 183, 91)	Yes	QR(367)
93	$(7 \cdot 53, 185, 92)$	No	$3^3 \equiv -1 \pmod{7}$
94	$(3 \cdot 5^3, 187, 93)$	No	$2 \equiv -1 \pmod{3}$
95	(379, 189, 94)	Yes	QR(379)
96	(383, 191, 95)	Yes	QR(383)
97	$(3^2 \cdot 43, 193, 96)$	No	97 would be multiplier
98	$(17 \cdot 23, 195, 97)$	No	$2^4 \equiv -1 \pmod{17}$
99	$(5 \cdot 79, 197, 98)$	No	$11 \equiv 3^{68} \pmod{395}$ would be multiplier
100	$(3 \cdot 7 \cdot 19, 199, 99)$	No	$4 = 2^2 \equiv 5^8 \pmod{399}$ would be multiplier

REFERENCES

- [Bar] BARKER, R. H. Group synchronizing of binary digital systems. In *Communication Theory*, W. Jackson, Ed., London: Butterworth, 1953, pp. 273-287.
- [Bau] BAUMERT, L. D. *Cyclic difference sets*. Lecture Notes in Mathematics 182, New York: Springer-Verlag, 1971.
- [BF] BAUMERT, L. D. and H. FREDRICKSEN. The Cyclotomic Numbers of Order Eighteen with Applications to Difference Sets. *Math. Comp.* 21 (1967), 204-219.
- [EKS] ELIAHOV, S., M. KERVAIRE and B. SAFFARI. A New Restriction on the Lengths of Golay Complementary Sequences. *J. Comb. Theory., Ser. A* 55 (1990), 45-59.
- [GMW] GORDON, B., W. H. MILLS and L. R. WELCH. Some New Difference Sets. *Canad. J. Math.* 14 (1962), 614-625.
- [H] HALL, M. Jr. *Combinatorial Theory*. Wiley-Interscience, Second Edition, 1986.
- [JL] JEDWAB, J. and S. LLOYD. A Note on the Nonexistence of Barker Sequences. *Designs, Codes and Cryptography* 2 (1992), 93-97.
- [L] LANDER, E. S. *Symmetric Designs: An Algebraic Approach*. London Math. Soc. Lecture Note Series 74, Cambridge University Press, 1983.