§1. Lemmas

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **38 (1992)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

first, and later we found out the fact that the classifying space is a join is the origin of rationality.

This paper is organized as follows. In § 1, we show two lemmas in algebraic topology. One asserts that the cup product is trivial on the cohomology ring of the join of two spaces. The other concerns the relationship between the tensor product in the E^2 term of the spectral sequence associated to the fibration $\Omega X \to PX \to X$ and the Pontrjagin product on the homology of ΩX. Both of them should be well known but we include their proofs. In § 2, we review the definition of discontinuous invariants of Morita ([10]). We see immediately that all higher discontinuous invariants vanish for codimension one transversely piecewise linear foliations. This implies the rationality of such foliations. The rest of this paper concerns the homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of the real line with compact support. This would be of interest because it would provide a good concrete example illustrating the relationship between the homology of the group of homeomorphisms and the homotopy of the classfying space for foliations. In § 3, we give the result of calculation of the homology of $PL_c(\mathbf{R})$. In § 4, we describe the way of calculation. This is done by defining sufficiently many cocycles. For this, we define and use a determinant with values in the tensor product over the rationals $\mathbf{Q}$ of a number of copies of $\mathbf{R}$. In § 5, we show the fact that the homomorphism $PL_c([0, \infty)) \to \mathbf{R}$ which sends f to $\log f'(0)$ induces a surjection in homology. Since there are no natural sections, this is not trivial. The nontriviality of the cocycles defined in § 4 depends on this fact.

My knowledge on the group of piecewise linear homeomorphisms of the real line was deepened during my visit à l'Université de Genève in the winter 1990/91. I would like to thank it for its warm hospitality. This work is done during my visit à l'Ecole Normale Supérieure de Lyon in the spring 1991. I would like to thank it for its warm hospitality and I also thank la Fondation Scientifique de Lyon et du Sud-Est for the financial support. I thank André Haefliger, Etienne Ghys, Peter Greenberg, Vlad Sergiescu and Shigeyuki Morita for their interest taken for this work.

§ 1. Lemmas

First we show the cup product is trivial on the cohomology ring of the join of two spaces. This is an exercise in algebraic topology.

Lemma (1.1). *Let X and Y be two topological spaces. The cup product on the cohomology ring of the join $X * Y$ is trivial.*

Proof. We may assume that X and Y are simplicial complexes. The simplices of the join $X*Y$ other than those in X and in Y are the joins of simplices of X and Y. Since $X*Y$ contains the cones of X and Y, any cocycle on $X*Y$ is cohomologous to a cocycle which vanishes on chains in X or Y. We look at the Alexander-Whitney approximation of the diagonal map $X*Y \to X*Y \times X*Y$. The image of a simplex in $X*Y$ in $C_*(X*Y) \otimes C_*(X*Y)$ is a sum of $\sigma_i \otimes \sigma_j$, where either σ_i or σ_j does not contain the edge corresponding to the joining interval. Hence the evaluation of the cup product of two modified cocycles is always zero.

The second lemma concerns the relationship between the tensor product in the E^2 term of the spectral sequence associated to the fibration $\Omega X \to PX \to X$ and the Pontrjagin product $*$ in the homology of the loop space ΩX.

LEMMA (1.2). *Let X be a simply connected CW complex such that $H_*(X;\mathbf{Z})$ is torsion free. Let PX and ΩX be the path space and the loop space of X, respectively. Let*

$$E^2_{p,q} = H_p(X;\mathbf{Z}) \otimes H_q(\Omega X;\mathbf{Z})$$

denote the E^2 term of the spectral sequence associated to the fibration. For positive integer p, there is a homomorphism

$$s\colon H_p(\Omega X;\mathbf{Z}) \to H_{p+1}(X;\mathbf{Z})$$

such that, for $v \in H_q(\Omega X;\mathbf{Z})$,

$$s(u) \otimes v \in E^2_{p+1,q} = H_{p+1}(X;\mathbf{Z}) \otimes H_q(\Omega X;\mathbf{Z})$$

and

$$u * v \in H_{p+q}(\Omega X;\mathbf{Z})$$

are related under ∂^{p+1}, where $*$ denotes the (Pontrjagin) product induced by the composition of loops. More precisely, for the submodules $Z^r_{p+1,q}$ and $B^r_{p+1,q}$ of $E^2_{p+1,q}$ which give $E^r_{p+1,q} = Z^r_{p+1,q}/B^r_{p+1,q}$,

$$s(u) \otimes v \in Z^p_{p+1,q} \quad \text{and} \quad \partial^{p+1}(s(u) \otimes v) - u * v \in B^p_{0,p+q}.$$

Proof. The element u is represented by the image of the fundamental class of a p-dimensional finite complex Y under a continuous map $Y \to \Omega X$. We define $s(u)$ to be the class represented by the adjoint map $SY \to X$, where SY denotes the suspension of Y. Since the composition $Y \to \Omega X \to PX$ bounds the map $SY \to PX$ in the obvious way and the composition $SY \to PX \to X$ represents $s(u) \in H_{p+1}(X;\mathbf{Z})$, $s(u)$ and u are related under ∂^{p+1}. Let $Z \to \Omega X$ represent v. Consider the composition

$$Y \times Z \to \Omega X \times \Omega X \xrightarrow{*} \Omega X .$$

Then this represents $u * v \in H_{p+q}(\Omega X; \mathbf{Z})$. On the other hand, the composition

$$Y \times Z \to \Omega X \times \Omega X \to PX \times \Omega X \to X \times \Omega X$$

bounds $SY \times Z \to PX \times \Omega X \to X \times \Omega X$, which represents $s(u) \otimes v$. Hence $s(u) \otimes v$ and $u * v$ are related under ∂^{p+1}.

§2. Discontinuous invariants

First we review the definition by Morita ([10]) of discontinuous invariants arising from the Godbillon-Vey invariant for codimension one foliations.

Let $\mathscr{F}$ be a codimension one foliation of a closed oriented $3k$-dimensional manifold M. Then the Godbillon-Vey class $gv(\mathscr{F}) \in H^3(M; \mathbf{R})$ is defined ([6]). Let $\{x_1, ..., x_n\}$ be a basis of $H^3(M; \mathbf{Q})$. Then $gv(\mathscr{F})$ is written as

$$gv(\mathscr{F}) = a_1 x_1 + \ldots + a_n x_n ,$$

where $a_1, ..., a_n \in \mathbf{R}$. The discontinuous invariant GV_k is defined by

$$GV_k(\mathscr{F}) = \sum_{i_1 < \ldots < i_k} (x_{i_1} \cup \ldots \cup x_{i_k})[M] \; a_{i_1} \wedge_{\mathbf{Q}} \ldots \wedge_{\mathbf{Q}} a_{i_k} \in \mathbf{R}^{\wedge k} = \overbrace{\mathbf{R} \wedge_{\mathbf{Q}} \ldots \wedge_{\mathbf{Q}} \mathbf{R}}^{k} ,$$

where $[M] \in H_{3k}(M; \mathbf{Z})$ is the fundamental class. Morita showed that GV_k is natural, GV_k depends only on the foliated cobordism class of $\mathscr{F}$, and hence there is a universal map $GV_k: H_{3k}(B\Gamma_1; \mathbf{Z}) \to \mathbf{R}^{\wedge k}$ ([10]).

The same argument applies to transversely piecewise linear foliations and the discrete Godbillon-Vey class defined in [5] and [3]. Then the following theorem is obtained from the description by Greenberg ([7]) of the classifying space for them and Lemma (1.1).

THEOREM (2.1). *Let $\mathscr{F}$ be a codimension one transversely orientable transversely piecewise linear foliation of a closed oriented $3k$-dimensional manifold $M (k \geqslant 2)$. Then $GV_k(\mathscr{F}) = 0$.*

Proof. The weak homotopy type of the classifying space $B\bar{\Gamma}_1^{PL}$ for codimension one transversely oriented transversely piecewise linear foliations is known by Greenberg ([7]). This classifying space $B\bar{\Gamma}_1^{PL}$ has the weak homotopy type of the join $B\mathbf{R}^\delta * B\mathbf{R}^\delta$ of two copies of $B\mathbf{R}^\delta = K(\mathbf{R}, 1)$. Let