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TOEPLITZ SEQUENCES, PAPERFOLDING, TOWERS OF HANOI AND PROGRESSION-FREE SEQUENCES OF INTEGERS

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ABSTRACT. What is the relationship between folding a piece of paper, moving disks in the classical tower of Hanoi algorithm and searching for minimal sequences of integers having no p terms in arithmetic progression? Our aim is to show how the Toeplitz sequences introduced by Jacobs and Keane in [15] allow us to give *(inter alia)* a unified description of the preceding problems. We give moreover some connections between Toeplitz sequences and q-automatic sequences.

1. TOEPLITZ SEQUENCES

In [15], (see also [21]), Jacobs and Keane defined the notion of Toeplitz sequence: they wanted to construct "explicit" sequences giving rise to strictly ergodic systems. They proved moreover that the unique invariant measure attached to such a sequence has a discrete rational spectrum. Roughly speaking a Toeplitz sequence is obtained by successive insertions of periodic sequences into the "holes" of a given periodic sequence, (a precise definition is given below). This construction was inspired by a device used by Toeplitz [28] for building explicitly almost periodic real functions. The method of Jacobs and Keane has since been used by many people working in ergodic theory (see for instance [29], [16] and [25], see also [14] and its impressive bibliography). We now give the definition of a Toeplitz sequence (compare with [15], [16], [14] and [29]):

Let $\Gamma = \{a_1, \cdots, a_r, \omega\}$ be an alphabet (finite set) with a "marked" letter ("hole") ω. If $B = (B(k))_{k \geqslant 0}$ is a sequence with values in Γ, we define a transformation $T_B: \Gamma^{\mathbf{N}} \to \Gamma^{\mathbf{N}}$ as follows: for any sequence $C = (C(k))_{k \geqslant 0}$ with values in Γ, let $h_0 < h_1 < \cdots$ be the increasing sequence (which might