§1. Murasugi's Congruence

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(See [Mi] or § 1 for definitions). We will be most interested in the case $F = \mathbf{F}_p$, the finite field with p elements.

THEOREM B. *Let G be a p-group. Suppose $C_\infty \times G$ act on a finite-dimensional* CW *complex X with* $\operatorname{rk} H_*(X; \mathbf{F}_p) < \infty$, *so that G acts semifreely and cellularly. Then*

$$\chi_m(X; \mathbf{F}_p)\chi_m(X^G; \mathbf{F}_p)^{|G|-1} = \chi_m(X/G; \mathbf{F}_p)^{|G|}.$$

Applying this to the case where X is the infinite cyclic cover of $\Sigma - K$ will immediately yield Murasugi's congruence. One advantage of our approach is that it generalizes to the case of high-dimensional periodic knots.

In § 1 we prove Theorem B and derive Theorem A. In § 2 we discuss the high-dimensional case and in § 3 give the following application of Murasugi's congruence to links.

PROPOSITION C. *Let L be a two-component link in a homology* 3-*sphere. If the* $\mathbf{Z}/2 \times \mathbf{Z}/2-$ *cover branched over the link is also a homology* 3-*sphere, then the linking number of the two components is congruent to* ± 1 *modulo* 8.

§ 1. MURASUGI'S CONGRUENCE

We will derive Theorem A from Theorem B and then prove Theorem B, but we first give some homological preliminaries. If R is a commutative Noetherian UFD with quotient field K and M is a finitely generated torsion R-module then we define the *order* of M to be $[M] = E^0(M) \in R/R^*$. Here we take an exact sequence

$$R^k \xrightarrow{A} R^m \rightarrow M \rightarrow 0,$$

and we let $E^0(M)$ be a greatest common divisor of the determinants of the $m \times m$-submatrices of A. If M is a torsion f.g. R-module then $[M] \neq 0$, and we consider the order $[M]$ as an element of K^*/R^*. If

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is an exact sequence of torsion f.g. R-modules, then J. Levine [L, lemma 5] shows $[M] = [M'] [M'']$. It follows for formal reasons that if $C_* = \{C_n \rightarrow \ldots \rightarrow C_0\}$ is a chain complex of torsion f.g. R-modules then

$$\chi_m(C_*) := \Pi[C_i]^{(-1)^i}$$

equals $\chi_m(H_*(C_*))$. In particular if C_* is exact, then $\chi_m(C_*) = 1$.

Next we turn to Alexander polynomials. By Alexander duality $H_1(\Sigma - K) \cong \mathbf{Z}$. Let $\pi: X \to \Sigma - K$ be the infinite cyclic cover of the knot complement. The infinite cyclic group $C_\infty = <t>$ acts on X and $H_1(X; \mathbf{Z})$ is a f.g. torsion module over the group ring $\mathbf{Z}[C_\infty] = \mathbf{Z}[t, t^{-1}]$. The Alexander polynomial $\Delta_K(t)$ is its associated order. (Note that $\mathbf{Z}[t, t^{-1}]^*$ consists of $\pm t^i$ and the quotient field of $\mathbf{Z}[t, t^{-1}]$ is the field of rational functions $\mathbf{Q}(t)$.) As usual we normalize so that $\Delta_K(t)$ is a polynomial with integer coefficients and non-zero constant term.

If K has period p^r, let $\bar{\pi}: \bar{X} \to \bar{\Sigma} - \bar{K}$ be the infinite cyclic cover of the quotient knot. The $G = \mathbf{Z}/p^r$-action on $\Sigma - K$ lifts to a G-action on X with quotient $\bar{X}$ and fixed set $\tilde{B} = \pi^{-1}(B)$. Indeed, let g be a generator of G. Then $g \circ \pi: X \to \Sigma - K$ induces the trivial map on H_1 and so lifts to $\tilde{g}: X \to X$. Since g has a non-empty, path-connected fixed-point set there is a unique lift $\tilde{g}$ with fixed points and the fixed point set is $\tilde{B}$. Since $\tilde{g}^{p^r}$ is a lift of the identity which has fixed points, it itself is the identity and hence $\tilde{g}$ is a map of period p^r. This gives an action of $C_\infty \times G$ on X. It further follows that $X/G \to \bar{\Sigma} - \bar{K}$ is an abelian cover inducing the trivial map on H_1, so that we can identify this cover with $\bar{\pi}$ and X/G with $\bar{X}$.

The cover π is classified by a map $c: \Sigma - K \to S^1 = K(\mathbf{Z}, 1)$ inducing an isomorphism on H_1. The inclusion map $B \to \Sigma - K$ induces multiplication by the linking number λ on H_1. Thus by considering $c|_B$ which classifies $\pi: \tilde{B} \to B$, we see $\tilde{B}$ is homeomorphic to λ disjoint copies of $\mathbf{R}$, cyclically permuted by the action of C_∞.

Now $H_i(X)$ and $H_i(\bar{X})$ are zero for $i > 1$ and $H_0(X)$ and $H_0(\bar{X})$ are isomorphic to $\mathbf{F}_p \cong \mathbf{F}_p[t, t^{-1}]/(t-1)\mathbf{F}_p[t, t^{-1}]$, so $\chi_m(X) = (t-1)/\Delta_K(t)$ and $\chi_m(\bar{X}) = (t-1)/\Delta_{\bar{K}}(t)$. Since $X^G = \tilde{B}$ consists of λ arcs cyclically permuted by $C_\infty = <t>$, $\chi(X^G) = t^\lambda - 1$. Putting this together with Theorem B we see

$$[(t-1)/\Delta_K(t)]\ [t^\lambda - 1]^{p^r - 1} = [(t-1)/\Delta_{\bar{K}}(t)]^{p^r}$$

or $\Delta_K(t) = \Delta_{\bar{K}}(t)^{p^r}(1 + t + \ldots + t^{\lambda - 1})^{p^r - 1}$ with the equality taking place in $\mathbf{F}_p(t)/\mathbf{F}_p[t, t^{-1}]^*$. This gives Murasugi's congruence.

Proof of Theorem B. We prove the theorem by induction on the order of G. Let G be a group of prime order p with generator g. Let

$$\sigma = 1 + g + g^2 + \dots + g^{p-1}$$
$$\delta = 1 - g$$

be elements of the group ring $\mathbf{F}_p[G]$. Note that $\delta\sigma = 0 = \sigma\delta$ and $\delta^{p-1} = \sigma$. We consider the following chain complexes of $\mathbf{F}_p[t, t^{-1}]$-modules (all homology is with $\mathbf{F}_p$-coefficients).

$$0 \to C_*(X^G) \to C_*(\bar{X}) \xrightarrow{\text{tr}} \sigma C_*(X) \to 0$$
$$0 \to \delta C_*(X) \oplus C_*(X^G) \to C_*(X) \xrightarrow{\sigma} \sigma C_*(X) \to 0$$
$$0 \to \sigma C_*(X) \to \delta C_*(X) \xrightarrow{\delta} \delta^2 C_*(X) \to 0$$
$$\vdots$$
$$0 \to \sigma C_*(X) \to \delta^{p-2} C_*(X) \xrightarrow{\delta} \delta^{p-1} C_*(X) \to 0 .$$

These induce long exact sequences in homology. All homology is finitely generated and torsion over the PID $\mathbf{F}_p[t, t^{-1}]$. We use shorthand notation – if $\rho \in \mathbf{F}_p[G]$, we write $\chi^\rho(X)$ instead of $\chi(H_*(\rho C_*(X)))$. The above homological considerations show

$$\chi(\bar{X}) = \chi(X^G)\chi^\sigma(X)$$
$$\chi(X) = \chi^\delta(X)\chi(X^G)\chi^\sigma(X)$$
$$\chi^\delta(X) = \chi^\sigma(X)\chi^{\delta 2}(X)$$
$$\vdots$$
$$\chi^{\delta^{p-2}}(X) = \chi^\sigma(X)\chi^\sigma(X) .$$

Multiplying all equations but the first together and cancelling terms we see

$$\chi(X) = \chi(X^G) \cdot \chi^\sigma(X)^p .$$

Using the first equation to substitute for $\chi^\sigma(X)$ one finds

$$\chi(X) = \chi(\bar{X})^p / \chi(X^G)^{p-1} .$$

Finally suppose G has order p^r. Let G_1 be a normal subgroup of index p. By the exact sequences above $\operatorname{rk} H_*(X/G_1; \mathbf{F}_p) < \infty$. By applying inductively the result for the G_1-action on X and the G/G_1 action on X/G_1, Theorem B follows.

§2. High-dimensional periodic knots

One advantage of our approach to Murasugi's congruence is that it applies equally well to a more general situation. Higher-dimensional periodic knots