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A knot K in a homology 3-sphere Σ has period n if it is invariant under a homeomorphism $h: \Sigma \to \Sigma$ of order exactly n with fixed set B, a circle disjoint from K. The quotient space $\bar{\Sigma} = \Sigma/h$ is a homology sphere containing $\bar{K}$, the quotient knot. Kunio Murasugi [Mu] discovered the following congruence involving the Alexander polynomials of the two knots. (See also the proof by J. Hillman [H].)

THEOREM A. *Let K be a knot of prime power period p^r in a homology 3-sphere Σ with fixed set B and quotient knot $\bar{K}$. Let $\Delta_K(t)$ and $\Delta_{\bar{K}}(t)$ be their Alexander polynomials and let λ be the linking number of K and B. Then*

$$\Delta_K(t) \doteq \Delta_{\bar{K}}(t)^{p^r}(1 + t + \dots + t^{\lambda - 1})^{p^r - 1} \pmod{p},$$

where $\doteq$ means congruent up to multiplication by ut^i where u and i are integers and u is relatively prime to p.

In another direction it is easily shown that if $G = \mathbf{Z}/p$ acts cellularly on a finite CW complex X, then $\chi(X) + (p-1)\chi(X^G) = p\chi(X/G)$. Using Smith theory, E. Floyd [F] gave a proof of this when X is a finite-dimensional CW complex with $\mathrm{rk}\, H_*(X; \mathbf{Z}/p) < \infty$. The proof can be generalized easily to the case of semifree actions of a p-group G on X. (An action is semifree if every point in X is either freely permuted by G or fixed by all of G. An action of $\mathbf{Z}/p$ is automatically semifree.) We will prove a multiplicative analogue of Floyd's theorem and use it to deduce Murasugi's congruence.

If X is a space with an action of the infinite cyclic group $C_\infty = \langle t \rangle$ and F is a field with $\mathrm{rk}\, H_*(X; F) < \infty$, we define a multiplicative Euler characteristic

$$\chi_m(X; F) \in F(t)^*/F[t, t^{-1}]^*$$

to be the alternating product of the generator of the order ideals of $H_i(X; F)$.

(See [Mi] or § 1 for definitions). We will be most interested in the case $F = \mathbf{F}_p$, the finite field with p elements.

THEOREM B. *Let G be a p-group. Suppose $C_\infty \times G$ act on a finite-dimensional* CW *complex X with* $\operatorname{rk} H_*(X; \mathbf{F}_p) < \infty$, *so that G acts semifreely and cellularly. Then*

$$\chi_m(X; \mathbf{F}_p)\chi_m(X^G; \mathbf{F}_p)^{|G|-1} = \chi_m(X/G; \mathbf{F}_p)^{|G|} .$$

Applying this to the case where X is the infinite cyclic cover of $\Sigma - K$ will immediately yield Murasugi's congruence. One advantage of our approach is that it generalizes to the case of high-dimensional periodic knots.

In § 1 we prove Theorem B and derive Theorem A. In § 2 we discuss the high-dimensional case and in § 3 give the following application of Murasugi's congruence to links.

PROPOSITION C. *Let L be a two-component link in a homology* 3-*sphere. If the* $\mathbf{Z}/2 \times \mathbf{Z}/2 -$ *cover branched over the link is also a homology* 3-*sphere, then the linking number of the two components is congruent to* ± 1 *modulo* 8.

§ 1. MURASUGI'S CONGRUENCE

We will derive Theorem A from Theorem B and then prove Theorem B, but we first give some homological preliminaries. If R is a commutative Noetherian UFD with quotient field K and M is a finitely generated torsion R-module then we define the *order* of M to be $[M] = E^0(M) \in R/R^*$. Here we take an exact sequence

$$R^k \xrightarrow{A} R^m \to M \to 0 ,$$

and we let $E^0(M)$ be a greatest common divisor of the determinants of the $m \times m$-submatrices of A. If M is a torsion f.g. R-module then $[M] \neq 0$, and we consider the order $[M]$ as an element of K^*/R^*. If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of torsion f.g. R-modules, then J. Levine [L, lemma 5] shows $[M] = [M'] \, [M'']$. It follows for formal reasons that if $C_* = \{C_n \to \ldots \to C_0\}$ is a chain complex of torsion f.g. R-modules then

$$\chi_m(C_*) := \Pi[C_i]^{(-1)^i}$$

equals $\chi_m(H_*(C_*))$. In particular if C_* is exact, then $\chi_m(C_*) = 1$.

Next we turn to Alexander polynomials. By Alexander duality $H_1(\Sigma - K) \cong \mathbf{Z}$. Let $\pi: X \to \Sigma - K$ be the infinite cyclic cover of the knot complement. The infinite cyclic group $C_\infty = <t>$ acts on X and $H_1(X;\mathbf{Z})$ is a f.g. torsion module over the group ring $\mathbf{Z}[C_\infty] = \mathbf{Z}[t, t^{-1}]$. The Alexander polynomial $\Delta_K(t)$ is its associated order. (Note that $\mathbf{Z}[t, t^{-1}]^*$ consists of $\pm t^i$ and the quotient field of $\mathbf{Z}[t, t^{-1}]$ is the field of rational functions $\mathbf{Q}(t)$.) As usual we normalize so that $\Delta_K(t)$ is a polynomial with integer coefficients and non-zero constant term.

If K has period p^r, let $\bar{\pi}: \bar{X} \to \bar{\Sigma} - \bar{K}$ be the infinite cyclic cover of the quotient knot. The $G = \mathbf{Z}/p^r$-action on $\Sigma - K$ lifts to a G-action on X with quotient $\bar{X}$ and fixed set $\tilde{B} = \pi^{-1}(B)$. Indeed, let g be a generator of G. Then $g \circ \pi: X \to \Sigma - K$ induces the trivial map on H_1 and so lifts to $\tilde{g}: X \to X$. Since g has a non-empty, path-connected fixed-point set there is a unique lift $\tilde{g}$ with fixed points and the fixed point set is $\tilde{B}$. Since $\tilde{g}^{p^r}$ is a lift of the identity which has fixed points, it itself is the identity and hence $\tilde{g}$ is a map of period p^r. This gives an action of $C_\infty \times G$ on X. It further follows that $X/G \to \bar{\Sigma} - \bar{K}$ is an abelian cover inducing the trivial map on H_1, so that we can identify this cover with $\bar{\pi}$ and X/G with $\bar{X}$.

The cover π is classified by a map $c: \Sigma - K \to S^1 = K(\mathbf{Z}, 1)$ inducing an isomorphism on H_1. The inclusion map $B \to \Sigma - K$ induces multiplication by the linking number λ on H_1. Thus by considering $c|_B$ which classifies $\pi: \tilde{B} \to B$, we see $\tilde{B}$ is homeomorphic to λ disjoint copies of $\mathbf{R}$, cyclically permuted by the action of C_∞.

Now $H_i(X)$ and $H_i(\bar{X})$ are zero for $i > 1$ and $H_0(X)$ and $H_0(\bar{X})$ are isomorphic to $\mathbf{F}_p \cong \mathbf{F}_p[t, t^{-1}]/(t-1)\mathbf{F}_p[t, t^{-1}]$, so $\chi_m(X) = (t-1)/\Delta_K(t)$ and $\chi_m(\bar{X}) = (t-1)/\Delta_{\bar{K}}(t)$. Since $X^G = \tilde{B}$ consists of λ arcs cyclically permuted by $C_\infty = <t>$, $\chi(X^G) = t^\lambda - 1$. Putting this together with Theorem B we see

$$[(t-1)/\Delta_K(t)]\ [t^\lambda - 1]^{p^r - 1} = [(t-1)/\Delta_{\bar{K}}(t)]^{p^r}$$

or $\Delta_K(t) = \Delta_{\bar{K}}(t)^{p^r}(1 + t + \ldots + t^{\lambda-1})^{p^r-1}$ with the equality taking place in $\mathbf{F}_p(t)/\mathbf{F}_p[t, t^{-1}]^*$. This gives Murasugi's congruence.

Proof of Theorem B. We prove the theorem by induction on the order of G. Let G be a group of prime order p with generator g. Let

$$\sigma = 1 + g + g^2 + \ldots + g^{p-1}$$
$$\delta = 1 - g$$

be elements of the group ring $\mathbf{F}_p[G]$. Note that $\delta\sigma = 0 = \sigma\delta$ and $\delta^{p-1} = \sigma$. We consider the following chain complexes of $\mathbf{F}_p[t, t^{-1}]$-modules (all homology is with $\mathbf{F}_p$-coefficients).

$$\begin{array}{l}
0 \to C_*(X^G) \to C_*(\bar{X}) \xrightarrow{\mathrm{tr}} \sigma C_*(X) \to 0 \\
0 \to \delta C_*(X) \oplus C_*(X^G) \to C_*(X) \xrightarrow{\sigma} \sigma C_*(X) \to 0 \\
0 \to \sigma C_*(X) \to \delta C_*(X) \xrightarrow{\delta} \delta^2 C_*(X) \to 0 \\
\qquad\vdots \\
0 \to \sigma C_*(X) \to \delta^{p-2} C_*(X) \xrightarrow{\delta} \delta^{p-1} C_*(X) \to 0 .
\end{array}$$

These induce long exact sequences in homology. All homology is finitely generated and torsion over the PID $\mathbf{F}_p[t, t^{-1}]$. We use shorthand notation – if $\rho \in \mathbf{F}_p[G]$, we write $\chi^\rho(X)$ instead of $\chi(H_*(\rho C_*(X)))$. The above homological considerations show

$$\begin{aligned}
\chi(\bar{X}) &= \chi(X^G)\chi^\sigma(X) \\
\chi(X) &= \chi^\delta(X)\chi(X^G)\chi^\sigma(X) \\
\chi^\delta(X) &= \chi^\sigma(X)\chi^{\delta^2}(X) \\
&\vdots \\
\chi^{\delta^{p-2}}(X) &= \chi^\sigma(X)\chi^\sigma(X) .
\end{aligned}$$

Multiplying all equations but the first together and cancelling terms we see

$$\chi(X) = \chi(X^G) \cdot \chi^\sigma(X)^p .$$

Using the first equation to substitute for $\chi^\sigma(X)$ one finds

$$\chi(X) = \chi(\bar{X})^p / \chi(X^G)^{p-1} .$$

Finally suppose G has order p^r. Let G_1 be a normal subgroup of index p. By the exact sequences above $\mathrm{rk}\, H_*(X/G_1; \mathbf{F}_p) < \infty$. By applying inductively the result for the G_1-action on X and the G/G_1 action on X/G_1, Theorem B follows.

§2. HIGH-DIMENSIONAL PERIODIC KNOTS

One advantage of our approach to Murasugi's congruence is that it applies equally well to a more general situation. Higher-dimensional periodic knots were introduced in the thesis of R. Cruz [C]. He showed that if there is a semifree $\mathbf{Z}/q$-action on S^n with non-empty fixed set and an invariant knot K^{n-2} disjoint from the fixed set, then the fixed set is S^1 if $q \neq 2$, and is S^1 or S^0 if $q = 2$.

For our purposes a knot K in a homology n-sphere Σ is an embedded $(n-2)$-dimensional homology sphere. Let G be a finite group. The knot K is *G-periodic* if it is invariant under a semifree G-action on Σ with fixed set $B \cong S^1$ disjoint from K. To simplify technicalities we assume the action is smooth. Several complications arise: the group need not be cyclic, the action need not be linear and the quotient $\bar{\Sigma} = \Sigma/G$ will not be a manifold. (Even in the linear case the quotient looks like a double suspension of a spherical space form.) However we can still make sense of Alexander polynomials.

PROPOSITION 2.1. $H_*(\bar{\Sigma} - \bar{K}) \cong H_*(S^1)$.

First we need a lemma.

LEMMA 2.2. *The linking number* $\lambda = \mathrm{lk}(B, K)$ *is relatively prime to the order of* G.

Proof. (See also [C, 2.1.1]). By restricting the action to a subgroup $\mathbf{Z}/p$ of G, we will assume $G = \mathbf{Z}/p$, and show $(\lambda, p) = 1$. By applying the Lefschetz Fixed-Point Theorem to a generator g of $\mathbf{Z}/p$, we see that if n is odd, the action on K is orientation-preserving, while if n is even, then $p = 2$ and the action is orientation-reversing. For local coefficients we will use $\mathbf{Z}^t$, the integers with the $\mathbf{Z}[\mathbf{Z}/p]$-module structure given by $(\Sigma a_i g^i) \cdot k = \Sigma a_i(-1)^{i(n+1)}k$.

Let $\bar{\Sigma} - B \to K(\mathbf{Z}/p, 1)$ classify the G-cover. We will consider the commutative diagram:

$$(*) \qquad \begin{array}{ccccc} H_{n-2}(K;\mathbf{Z}) & \xrightarrow{\alpha} & H_{n-2}(\bar{K};\mathbf{Z}^t) & \to & H_{n-2}(K(\mathbf{Z}/p,1);\mathbf{Z}^t) \\ \downarrow & & \downarrow & & \| \\ H_{n-2}(\Sigma - B;\mathbf{Z}) & \to & H_{n-2}(\bar{\Sigma} - B;\mathbf{Z}^t) & \to & H_{n-2}(K(\mathbf{Z}/p,1);\mathbf{Z}^t) \end{array}.$$

The two groups on the left are infinite cyclic and the left vertical map is multiplication by λ. A diagram chase shows we will be done if we can show both horizontal exact sequences are isomorphic to the short exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/p \to 0$.

The map α is isomorphic to $\mathbf{Z} \xrightarrow{\times p} \mathbf{Z}$ because it comes from a p-fold cover of $(n-2)$-dimensional closed manifolds. The map

$$H_{n-2}(\bar{K};\mathbf{Z}^t) \to H_{n-2}(\mathbf{Z}/p;\mathbf{Z}^t)$$

we compute algebraically by using a free $\mathbf{Z}G$-resolution of $\mathbf{Z}$ as a substitute for the Eilenberg-MacLane space. By lifting a CW structure on $\bar{K}$ to K,

$$C_*(K) = \{C_{n-2} \to \ldots \to C_0\}$$

with the i-chains C_i free $\mathbf{Z}G$-modules. By mapping a free $\mathbf{Z}G$-module onto $\ker(C_{n-2} \to C_{n-3})$ and continuing inductively, one constructs a free $\mathbf{Z}G$-resolution of $\mathbf{Z}$

$$D_* = \{\ldots \to D_n \to D_{n-1} \to C_{n-2} \to \ldots \to C_0\}\ .$$

It follows that

$$H_{n-2}(\bar{K}; \mathbf{Z}^t) = H_{n-2}(C_*(K) \otimes_{\mathbf{Z}G} \mathbf{Z}^t)$$

maps onto $H_{n-2}(D_* \otimes_{\mathbf{Z}G} \mathbf{Z}^t) = H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t)$. Furthermore by using the standard $\mathbf{Z}G$-resolution of $\mathbf{Z}$ (see e.g. [Mac]), one easily computes that $H_{n-2}(\mathbf{Z}/p; \mathbf{Z}^t) \cong \mathbf{Z}/p$.

Choose a G-invariant normal disk to B in Σ and let S^{n-2} be its boundary. Then the inclusion $S^{n-2} \to \Sigma - B$ is a homology equivalence. By the comparison theorem applied to the spectral sequence of the G-coverings (see [Mac]), the bottom row of (*) is isomorphic to

$$H_{n-2}(S^{n-2}; \mathbf{Z}) \to H_{n-2}(S^{n-2}/G; \mathbf{Z}^t) \to H_{n-2}(G; \mathbf{Z}^t)\ ,$$

and hence by the previous paragraph to $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/p \to 0$. Thus $(\lambda, p) = 1$.

Proof of 2.1. Let N be an equivariant tubular neighborhood of B. Then

$$0 = H_*(\Sigma - K, N; \mathbf{Z}[1/\lambda]) = H_*(\Sigma - K - B, N - B; \mathbf{Z}[1/\lambda])$$

where the first equality holds by the definition of linking number and the second by excision. Then

$$\begin{aligned} 0 = H_*((\Sigma - K - \mathrm{B})/G, (N - B)/G; \mathbf{Z}[1/\lambda]) &= H_*((\Sigma - K)/G, N/G; \mathbf{Z}[1/\lambda]) \\ &= H_*((\Sigma - K)/G, B; \mathbf{Z}[1/\lambda])\ , \end{aligned}$$

where the first equality follows from the spectral sequence of a covering, the second by excision and the third by the homotopy equivalence $B \to N/G$. Thus $H_*(\bar{\Sigma} - \bar{K})$ looks like $H_*(S^1)$ except possibly for some λ-torsion. But by 2.1, λ is prime to the order of G, so for all primes q dividing λ, the transfer map $\mathrm{tr}: H_*(\bar{\Sigma} - \bar{K}; \mathbf{Z}/q) \to H_*(\Sigma - K; \mathbf{Z}/q)$ is injective so there is no extra λ-torsion.

To state Murasugi's congruence in higher dimensions is it necessary to find a substitute for the Alexander polynomial. Let X and $\bar{X}$ be the infinite cyclic covers of $\Sigma - K$ and $\bar{\Sigma} - \bar{K}$ respectively. Let $\Delta_K(t) = \Pi_{i>0}[H_i(X)]^{(-1)^{i+1}}$ and $\Delta_{\bar{K}}(t) = \Pi_{i>0}[H_i(\bar{X})]^{(-1)^{i+1}}$. The Wang sequence shows that multiplication by $t-1$ induces an isomorphism on $H_i(X)$ for $i > 0$, so that if we take the polynomial represented by $[H_i(X)]$ and plug in $t = 1$ we get ± 1. (Indeed if we consider the ring homomorphism $\varphi : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z}$ defined by $\varphi(t) = 1$, then $\varphi([H_i(X)])$ is a divisor of $[H_i(X) \otimes_{\mathbf{Z}[t,t^{-1}]} \mathbf{Z}] = [0] = 1 \in \mathbf{Z}/\mathbf{Z}^*$.) Thus $[H_i(X)]$ represented a non-zero element in $\mathbf{F}_p[t, t^{-1}]$, and hence $\Delta_K(t)$ and $\Delta_{\bar{K}}(t)$ give well-defined elements of $\mathbf{F}_p(t)^*/\mathbf{F}_p[t, t^{-1}]^*$. Then the considerations of §1 show:

THEOREM 2.3. *Let K be a G-periodic knot in a homology q-sphere Σ with fixed set B, where G is a group of prime power order p^r. Let λ be the linking number of K and B. Then*

$$\Delta_K(t) \doteq \Delta_{\bar{K}}(t)^{p^r}(1 + t + \ldots + t^{\lambda-1})^{p^r - 1} \pmod{p} .$$

§3. AN APPLICATION OF MURASUGI'S CONGRUENCE

For any $\lambda \equiv \pm 1 \pmod 8$, T. tom Dieck and J. Davis [D-D] constructed a 2-component link with linking number λ in a homology 3-sphere Ω whose $C_2 \times C_2$-cover branched over the link is a homology 3-sphere Σ. We will show that this congruence condition is necessary. Equivalently, we show

THEOREM 3.1. *Suppose the Klein 4-group $G \times H \cong C_2 \times C_2$ acts on a homology 3-sphere Σ so that the fixed sets Σ^G and Σ^H are disjoint circles. Then their linking number λ is congruent to ± 1 modulo 8.*

Proof. We have

$$\begin{array}{ccc} \Sigma & \to & \Sigma/G \\ \downarrow & & \downarrow \\ \Sigma/H & \to & \Sigma/(G \times H) . \end{array}$$

All four of these manifolds are homology 3-spheres and each has two disjoint circles given by the images of the fixed sets. The linking numbers of each pair of circles are all equal.

Let $K = \Sigma^G/G \subset \Sigma/G$ and $\bar{K} = K/H \subset \Sigma/(G \times H)$. Then K is a knot of period 2. Renormalize $\Delta_K(t)$ and $\Delta_{\bar{K}}(t) \in \mathbf{Z}[t, t^{-1}]$ so that $\Delta_K(t) = \Delta_K(t^{-1})$, $\Delta_{\bar{K}}(t) = \Delta_{\bar{K}}(t^{-1})$, and $\Delta_K(1) = 1 = \Delta_{\bar{K}}(1)$. Murasugi's congruence shows

$$(**) \qquad \Delta_K(t) = \Delta_{\bar{K}}(t)^2(t^{(1-\lambda)/2} + \ldots + 1 + \ldots + t^{(\lambda-1)/2}) + 2f(t)\,,$$

where $f(t) \in \mathbf{Z}[t, t^{-1}]$ satisfies $f(t) = f(t^{-1})$. Writing

$$f(t) = a_n t^{-n} + \ldots + a_0 + \ldots + a_n t^n\,,$$

we see $f(1) \equiv f(-1) \pmod 4$. Since $\Sigma \to \Sigma/G$ is a 2-fold cover branched over K, $|\Delta_K(-1)| = |H_1(\Sigma)| = 1$. So $1 = \Delta_K(1) \equiv \Delta_K(-1) \pmod 4$, and we see $\Delta_K(-1) = 1$. Take equation (**) and plug in $t = 1$ and $t = -1$:

$$1 = 1 \cdot \lambda + 2 \cdot f(1)$$
$$1 = 1 \cdot (-1)^{(\lambda-1)/2} + 2 \cdot f(-1)\,.$$

Thus $\lambda \equiv (-1)^{(\lambda-1)/2} \pmod 8$ so $\lambda \equiv \pm 1 \pmod 8$.

Applying the high-dimensional version of Murasugi's congruence ones sees that if $G \times H \cong C_2 \times C_2$ acts on a homology q-sphere Σ so that Σ^G is a homology $q-2$ sphere and Σ^H is a circle disjoint from Σ^G, then their linking number λ is congruent to ± 1 modulo 8. This and considerations from L-theory lead us to conjecture that if $G \times H \cong C_2 \times C_2$ acts on a homology q-sphere Σ so that Σ^G is a homology k-sphere and Σ^H is a homology $q-k-1$-sphere disjoint from Σ^G, then their linking number λ is congruent to ± 1 modulo 8.

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REFERENCES

[C] da CRUZ, R. N. *Periodic knots.* Thesis (New York University 1987).

[D-D] DAVIS, J. F. and T. TOM DIECK. Some exotic dihedral actions on spheres. *Indiana Univ. Math. J. 37* (1988), 431-450.

[F] FLOYD, E. On periodic maps and the Euler characteristics of associated spaces. *Trans. Amer. Math. Soc. 72* (1952), 138-147.

[H] HILLMAN, J. A. New proofs of two theorems on periodic knots. *Arch. Math. (Basel) 37* (1981), 457-461.

[L] LEVINE, J. A method for generating link polynomials. *Amer. J. Math. 89* (1967), 69-84.

[Mac] MAC LANE, S. *Homology* (Springer-Verlag 1963).

[Mi] MILNOR, J. Infinite cyclic covers. *Conf. on topology of manifolds,* ed. J. G. Hocking (Prindle, Weber, and Schmidt, 1968), 115-133.

[Mu] MURASUGI, K. On periodic knots. *Comment. Math. Helv. 46* (1971), 162-174.

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