1. Introduction

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COHOMOLOGY OF FINITELY GENERATED ABELIAN GROUPS

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1. INTRODUCTION

Let Q be a finitely generated abelian group. In this note we compute its integral cohomology explicitly. Our approach is to construct a suitable small model $\mathscr{A}(Q)$ for the algebra of cochains on Q. Our model is very simple and relies on some differential homological algebra that has been created more than three decades ago [2-4], [8], [9], [28], [29]. Here is the model:

THEOREM A. *With coefficients in an arbitrary commutative* R, *the cohomology ring of a finitely generated abelian group* Q, *given by a presentation*

$$Q = < f_1, \ldots, f_m, t_1, \ldots, t_n \; ; \; t_j^{l_j} = 1 > ,$$

is as an R-algebra isomorphic to the homology algebra of the following primitively generated cocommutative differential graded Hopf algebra $\mathscr{A}(Q)$:

Generators: $z_1, \ldots, z_m, |z_j| = 1, x_1, \ldots, x_n, |x_i| = 1, c_1, \ldots, c_n, |c_i| = 2.$

Relations: *The generators commute in the graded sense, save that the squares of the* x_i*'s are non-zero and given by*

$$x_i^2 = \frac{l_i(l_i - 1)}{2} c_i, 1 \leqslant i \leqslant n \; ; \tag{1.1}$$

in particular the squares of the z_j*'s are zero.*

Differential: $d(z_j) = 0, 1 \leqslant j \leqslant m, \quad d(x_i) = l_i c_i, \quad d(c_i) = 0, 1 \leqslant i \leqslant n.$

For appropriate coefficients so that the diagonal map of this Hopf algebra induces a diagonal map on its homology, e.g. if R *is a field or if the characteristic of* R *divides each* $l_i, 1 \leqslant i \leqslant n,$ *the isomorphism is one of Hopf algebras.*

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Here as usual $|w|$ denotes the degree of a homogeneous element w. A proof of this Theorem has been given in Section 3 of our paper [13]. To make the present paper self contained we reproduce the proof in Section 2 below.

The model philosophy has already succesfully been exploited by us in [13-20] for theoretical purposes and to do calculations in group cohomology and algebraic topology. Models of different kinds have been introduced and exploited in the literature for decades, see for example [1-3], [9-11], [21], and the literature there; more about e.g. rational homotopy theory and minimal models may be found in Halperin's book [11].

Our main aim is to deduce an explicit description of the integral cohomology ring of Q from Theorem A. To explain our result, we denote an *infinite* cyclic group by C and a *finite* cyclic group of order l by C_l. We then write our abelian group Q in the form $Q = C_{l_1} \times \cdots \times C_{l_n} \times C^m$. It is clear that, as a graded commutative algebra, the integral cohomology ring $H^*(Q, \mathbf{Z})$ decomposes as

$$H^*(Q, \mathbf{Z}) = H^*(C_{l_1} \times \cdots \times C_{l_n}, \mathbf{Z}) \otimes H^*(C^m, \mathbf{Z}) \ . \tag{1.2}$$

Moreover, cf. Eckmann [7],

$$H^*(C^m, \mathbf{Z}) = \Lambda[z_1, \cdots, z_m] \tag{1.3}$$

where the notation z_j is abused somewhat. We therefore assume henceforth that Q is finite. Moreover, we suppose that things have been arranged in such a way that

$$l_1 \mid l_2 \mid \ldots \mid l_n \ , \tag{1.4}$$

where as usual '$u \mid v$' means that the number u divides the number v. It is classical that this is always possible. From Theorem A we deduce at once that every monomial of the kind

$$x_i x_j \ldots x_k, \qquad 1 \leqslant i < j < \cdots < k \leqslant n \ , \tag{1.5}$$

is a cycle in our algebra $\mathscr{A}(Q)$ when taken over the ground ring $R = \mathbf{Z}/l_i$ and hence determines a cohomology class $x_i x_j \ldots x_k \in H^*(-, \mathbf{Z}/l_i)$, where a slight abuse of notation comes into play. For each l_i, let

$$\beta_{l_i}\colon H^*(-, \mathbf{Z}/l_i) \to H^{*+1}(-, \mathbf{Z}) \tag{1.6}$$

be the indicated Bockstein operation and, for each monomial of the kind (1.5), let

$$\zeta_{x_i x_j \ldots x_k} = \beta_{l_i}(x_i x_j \ldots x_k) \in H^{*+1}(Q, \mathbf{Z}) \ . \tag{1.7}$$

It is clear that, in our model $\mathscr{A}(Q)$, the class $\zeta_{x_i x_j \ldots x_k}$ is represented by the cocycle

$$(1.8)\qquad \frac{1}{l_i} d(x_i x_j \ldots x_k) = c_i x_j \ldots x_k - \frac{l_j}{l_i} c_j x_i \ldots x_k + \cdots \pm \frac{l_k}{l_i} c_k x_i x_j \ldots x_{k-1} .$$

Moreover, inspection of the Bockstein exact sequence

$$(1.9)\qquad H^*(Q, \mathbf{Z}) \xrightarrow{l_i} H^*(Q, \mathbf{Z}) \to H^*(Q, \mathbf{Z}/l_i) \xrightarrow{\beta_{l_i}} H^{*+1}(Q, \mathbf{Z})$$

shows that the class $\zeta_{x_i x_j \ldots x_k} \in H^{*+1}(Q, \mathbf{Z})$ has exact order l_i.

We now consider the graded algebra $A(Q)$ that arises from $\mathscr{A}(Q)$ by introducing the additional relations $l_i c_i = 0$. It is clear that, when we write ζ_i for the obvious image of $c_i \in \mathscr{A}(Q)$ in $A(Q)$, as a graded algebra, $A(Q)$ is generated by

$$(1.10.1)\qquad x_1, \ldots, x_n, |x_i| = 1 , \zeta_1, \ldots, \zeta_n, |\zeta_i| = 2 ,$$

subject to the relations that

(1.10.2) the generators commute in the graded sense, save that the squares of the x_i's are possibly non-zero and given by

$$x_i^2 = \frac{l_i(l_i - 1)}{2} \zeta_i, 1 \leqslant i \leqslant n ;$$

and

$$(1.10.3)\qquad l_i \zeta_i = 0, 1 \leqslant i \leqslant n .$$

We note that when l_i is odd we have in fact $x_i^2 = 0$. By construction, the association

$$c_i \mapsto \zeta_i , \qquad x_j \mapsto x_j$$

yields a surjective morphism

$$(1.11)\qquad \mathscr{A}(Q) \to A(Q)$$

of differential graded algebras where the algebra $A(Q)$ is understood to have zero differential. Since (1.11) is an isomorphism in degree 1 we do not distinguish in notation between $x_j \in \mathscr{A}(Q)$ and its image in $A(Q)$. For each monomial $x_i x_j \ldots x_k$ of the kind (1.5), we write

$$\tilde{\zeta}_{x_i x_j \ldots x_k} \in A(Q)$$

for the image of $\zeta_{x_i x_j \ldots x_k} \in \mathscr{A}(Q)$ under (1.11), so that

$$(1.12)\quad \tilde{\zeta}_{x_i x_j \ldots x_k} = \zeta_i x_j \ldots x_k - \frac{l_j}{l_i} \zeta_j x_i \ldots x_k + \cdots \pm \frac{l_k}{l_i} \zeta_k x_i x_j \ldots x_{k-1} \in A(Q) .$$

Here is our main result.

THEOREM B. *For a finite abelian group* Q, *the association*

$$(1.13)\quad \zeta_{x_i x_j \ldots x_k} \mapsto \tilde{\zeta}_{x_i x_j \ldots x_k} \in A(Q)$$

identifies $H^*(Q, \mathbf{Z})$ *with the graded subalgebra of* $A(Q)$ *generated (as an algebra) by the* $\tilde{\zeta}_{x_i x_j \ldots x_k} \in A(Q)$.

This description of the cohomology ring should be compared with that in Chapman [5]. Our description of the cohomology ring has its advantages and disadvantages. An advantage is that it arises from a "small" model; indeed, modulo a prime p dividing each torsion coefficient l_i, our model boils down to the cohomology ring itself. To have a model as small as possible is important for explicit computations. A disadvantage of our description is that it is natural in the *presentation only,* and not in the group itself. Invariant descriptions of the *homology* of a finitely generated abelian group have been given by Hamsher [12] and Decker [6] in their Chicago ph. d. theses supervised by S. Mac Lane. We do not known whether invariant descriptions of the *cohomology* of a finitely generated abelian group have ever been worked out.

I am indebted to S. Mac Lane for discussions and for a number of comments about an earlier version of the paper.

2. THE PROOF OF THEOREM A

To make the paper self contained we reproduce the following material from our paper [13]. The contents of the present Section are of course classical but apparently not as well known as they deserve.

Let R be a commutative ring with 1. By a *Hopf algebra* $(H, \mu, \Delta, \eta, \varepsilon)$ over R we mean as usual a module H together with the structures (μ, η) and (Δ, ε) of an algebra and a coalgebra that are compatible, that is, μ is a morphism of coalgebras or, equivalently, Δ is a morphism of algebras; see e.g. VI.9 in Mac Lane [23]. We mention in passing that some authors call this a bialgebra and require a Hopf algebra to have the additional structure of what is called an *antipode.* For us this is actually of no account since this additional structure will always be present, but there is no need to spell it out. Examples of Hopf algebras are group rings, exterior Hopf algebras, divided