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Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

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7. Pompeiu property for two-sided translations on groups

In this section we consider the Pompeiu problem in the setting where X is one of the groups $M(2)$ or $SL(2, \mathbf{R})$ and the group action is that of two-sided translations. We start with a general definition.

Definition 7.1. Let G be a locally compact unimodular group. A measurable relatively compact subset $E \subseteq G$ of positive Haar measure is said to have the (two-sided) Pompeiu property if for $f \in C(G)$,

$$\int_{g_1 E g_2} f(x)dx = 0 \quad \forall g_1, g_2 \in G$$

implies $f \equiv 0$ where the integration is with respect to the Haar measure.

Except for the two groups mentioned the problem in general appears to be untractable at the moment, the primary reason being that there has not been much progress with the related problem of two-sided spectral analysis. If, however, we consider only functions $f \in L^1(G) \cap C(G)$ the problem considered above becomes easier and some investigations have been made in this restricted set-up (see [20], [21], [24]). Our treatment of $M(2)$ and $SL(2, \mathbf{R})$ relies on the work of Weit ([32]) and Ehrenpreis and Mautner ([13], [14]) on spectral analysis and synthesis on $M(2)$ and $SL(2, \mathbf{R})$ respectively. (See also [4], [31] in this connection.)

§ 7.1. We introduce a class of representations of the group $M(2)$. As in Section 3, we write $M(2) = \{(x, \sigma) : x \in \mathbf{R}^2, \sigma \in SO(2, \mathbf{R})\}$ where $(x, \sigma) \cdot (x', \sigma') = (x + \sigma x', \sigma\sigma')$ is the group multiplication and an element (x, σ) acts on $y \in \mathbf{R}^2$ by the rule $(x, \sigma)y = \sigma y + x$. Let $\lambda \in \mathbf{C}, \lambda \neq 0$ and we define representations Π_λ on the space $L^2(S^1)$ where $S^1 \subseteq \mathbf{R}^2$ is the unit circle: for $(x, A) \in M(2)$,

$$\Pi_\lambda(x, A)f(w) = e^{i\lambda x \cdot w} f(A^{-1}w) \quad f \in L^2(S), \quad w \in S^1,$$

where $\cdot$ is the inner product in $\mathbf{R}^2$. The Π_λ's are related to the representations of $M(2)$ on the eigenspaces of the Laplacian Δ on $\mathbf{R}^2$ and are irreducible (see [16], p. 12). If λ is real, then Π_λ is seen to be a unitary representation. The only other irreducible unitary representation (up to equivalence) of $M(2)$ are the characters:

$$\chi_n(x, A) = e^{in\theta}, \quad x \in \mathbf{R}^2, A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where $n = 0, \pm 1, \pm 2, \cdots$.

We now come to our result for the group $M(2)$.

THEOREM 7.1. *A measurable relatively compact subset* $E \subseteq M(2)$ *of positive Haar measure has the Pompeiu property if and only if*

(i) *the operators* $\int_E \Pi_\lambda(x)dx \neq 0$ *for each* $\lambda \in \mathbf{C}, \lambda \neq 0$;

(ii) $\int_E \chi_n(x)dx \neq 0$ *for each integer* n;

where the integrals are with respect to the Haar measure on $M(2)$.

Proof. If Π is a continuous irreducible representation of G on a Hilbert space and $\Pi(1_E) = 0$, then for a suitably chosen matrix element $f: x \to \langle \Pi(x)v_1, v_2 \rangle$, we have $\int_{g_1 E g_2} f(x)dx = 0$ for all g_1, g_2. The *only if* part now follows. To prove the *if* part we consider the two-sided ideal of $C(M(2))$:

$$\mathcal{U} = \{f \in C(M(2)) : \int_{g_1 E g_2} f dx = 0 \quad \text{for all} \quad g_1, g_2 \in M(2)\}.$$

Assume $\mathcal{U} \neq \{0\}$. We shall prove that either for some $n, \int_E \chi_n(x)dx = 0$, or for some $\lambda \neq 0, \int_E \Pi_\lambda(x)dx = 0$. Since $\mathcal{U} \neq \{0\}$, by Weit's theorem ([32], Theorem 1), either $\chi_n \in \mathcal{U}$ for some n or there exists $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2, \lambda_1^2 + \lambda_2^2 \neq 0$ such that the functions $g_\lambda(x, A) = e^{i(\lambda_1 x_1 + \lambda_2 x_2)}$, where $(x, A) \in M(2)$ $(x = (x_1, x_2))$, belongs to $\mathcal{U}$. In case $\chi_n \in \mathcal{U}$ we immediately get the desired result. So let for λ as above, $g_\lambda \in \mathcal{U}$. As noted by Weit if $\mu = (\mu_1, \mu_2) \in \mathbf{C}^2$ and $\mu_1^2 + \mu_2^2 = \lambda_1^2 + \lambda_2^2$, then $g_\mu \in \mathcal{U}$. We choose $\mu_1 = \sqrt{\lambda_1^2 + \lambda_2^2} \cos\theta, \mu_2 = \sqrt{\lambda_1^2 + \lambda_2^2} \sin\theta$ for some $\theta \in \mathbf{R}$, so that

$$g_\mu(x, A) = \exp(i\sqrt{\lambda_1^2 + \lambda_2^2}\, x \cdot w), \quad (x, A) \in M(2),$$

where $w = (\cos\theta, \sin\theta)$. Since $g_\mu \in \mathcal{U}$ for all $\theta \in \mathbf{R}$ and since $\mathcal{U}$ is closed, we have $h \in \mathcal{U}$ where

$$h(x, A) = \int_{S^1} \exp(i\sqrt{\lambda_1^2 + \lambda_2^2}\, x \cdot w) dw, \quad (x, A) \in M(2).$$

But $h(x, A) = \langle \Pi_z(x, A)1, 1 \rangle, (x, A) \in M(2)$ where $z = \sqrt{\lambda_1^2 + \lambda_2^2}$ and so we have

$$\int_{g_1 E g_2} \langle \Pi_z(g)1, 1 \rangle \, dg = 0$$

for all $g_1, g_2 \in G$. This means

$$\int_E <\Pi_z(g_1)\Pi_z(g)\Pi_z(g_2)1, 1> dg = 0, \quad g_1, g_2 \in G.$$

By taking adjoints and using the fact $\Pi_z(g)^* = \Pi_{-\bar{z}}(g^{-1})$,

$$\int_E <\Pi_z(g)\Pi_z(g_2)1, \Pi_{-\bar{z}}(g_1^{-1})1> dg = 0, \quad g_1, g_2 \in G.$$

Since Π_z and $\Pi_{-\bar{z}}$ are irreducible, 1 is a cyclic vector in $L^2(S^1)$ for both these representation and it will, therefore, follow

$$\int_E <\Pi_z(g)u, v> dg = 0$$

for all $u, v \in L^2(S^1)$. This shows that $\Pi_z(1_E) = 0$ and the theorem is proved.

§ 7.2. For simplicity, we consider the group

$$PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\{+1, -1\}$$

rather than $SL(2, \mathbf{R})$ itself. For each $\lambda \in \mathbf{C}$, $SL(2, \mathbf{R})$ has the so-called principal series representation Π_λ (which are continuous representations realised on a Hilbert space). It is well known, that but for a countable set of λ's Π_λ is irreducible. Thus the set of representations $\{\Pi_\lambda : \lambda \in \mathbf{C}, \Pi_\lambda \text{ irreducible}\}$ along with another countable family of representations called the discrete series and all irreducible finite-dimensional representations account for all so-called "topologically completely irreducible" Banach representations (upto equivalence) of $PSL(2, \mathbf{R})$. We now state our theorem for $PSL(2, \mathbf{R})$:

THEOREM 7.2. *A relatively compact measurable subset* $E \subseteq PSL(2, \mathbf{R})$ *of positive Haar measure has the Pompeiu property if and only if for each topologically completely irreducible Banach representation* Π *of* $PSL(2, \mathbf{R})$, $\Pi(1_E) \neq 0$.

Sketch of Proof. As with Theorem 7.1, the proof relies on spectral analysis of two-sided ideals of $C^\infty(G)$ due to Ehrenpreis and Mautner ([13], [14]): If $0 \neq \mathscr{U} \subseteq C^\infty(G)$ is a closed two-sided ideal in $C^\infty(G)$, then there exists $\lambda \in \mathbf{C}$ and a nontrivial function of the form $\phi^\lambda_{n,m} : x \to <\Pi_\lambda(x)e_n, e_m>$ is in $\mathscr{U}$. Here $\{e_j\}$ is an orthonormal basis of $\mathscr{H}_\lambda$, the representation space for Π_λ and moreover, $\Pi_\lambda(k)e_j = \chi_j(k)e_j$ where $\{\chi_j\}$ are the characters of the maximal compact subgroup $SO(2, \mathbf{R})/\{+1, -1\}$ of $PSL(2, \mathbf{R})$ and $k \in SO(2, \mathbf{R})/\{+1, -1\}$. To prove the theorem, we define $\mathscr{U}$ as in Theorem 7.1. If $\mathscr{U} \neq \{0\}$, then appealing to the result quoted above we get $\phi^\lambda_{n,m} \in \mathscr{U}$ for some $\lambda \in \mathbf{C}$ and n, m integers. If the corresponding Π_λ is irreducible, then the argument is as in Theorem 7.1 and we can prove $\Pi_\lambda(1_E) = 0$. If Π_λ is not irreducible, then depending on n, m we can find a discrete series representation or an irreducible finite dimensional representation Π occurring either as a subrepresentation or as a subquotient of Π_λ for which $\Pi(1_E) = 0$. To do this, we need the exact G-module structure of Π_λ which in the case of $PSL(2, \mathbf{R})$ is available (see for example [14]).

§ 7.3. Consider the case $G = \mathbf{R}^2$ and G acting on itself *by translations*. In this case, Brown, Schreiber and Taylor have proved that there are no Pompeiu sets ([12]). In view of this it would be natural to ask if there are sets E satisfying the conditions of Theorem 7.1 at all. Identify $\mathbf{R}^2$ with G/K where $G = M(2)$ and $K = SO(2, \mathbf{R})$; if $E \subseteq G/K$ then one can show that the condition of Brown, Schreiber and Taylor considered in Section 3 is equivalent to the condition $\Pi_\lambda(1_{\tilde{E}}) \neq 0$ for $\lambda \in \mathbf{C}, \lambda \neq 0$. (A special case of this observation is also made in [30]). Hence by the discussion in Section 3, there are plenty of sets E with this property. As we have seen, topologically $G \approx \mathbf{R}^2 \times SO(2, \mathbf{R})$. We now observe that if E is chosen as above in $\mathbf{R}^2$ and I is a suitably chosen arc in $SO(2, \mathbf{R})$, then the set $F = E \times I$ considered as a subset of G satisfies $\Pi_\lambda(1_F) \neq 0$ for all $\lambda \in \mathbf{C} - \{0\}$ as well as $\chi_n(1_F) \neq 0$ for all $n \in \mathbf{Z}$. (We need only to choose I whose length is irrational modulo 2π.)

8. Concluding remarks

In this paper, we have restricted our attention to the Pompeiu property for a single set E. One can also consider the Pompeiu property for a collection of sets or distributions of compact support as in [9], [12]. There are also closely related properties such as the Morera property — see [12] for details.

As pointed out earlier the Pompeiu problem becomes easier if one considers only integrable functions. Investigations under this assumption have been done, for example, in [2], [20], [24] and [28]. If one only considers integrable functions one need not restrict oneself to relatively compact sets. Moreover, considering integrable functions is equivalent to considering finite complex measures. Thus for G a locally compact abelian group a Borel subset $E \subseteq G$ is said to be a *determining set for finite complex measures* if for a finite complex measure μ on G, $\mu(gE) = 0$ for all $g \in G$ implies $\mu = 0$.

For locally compact abelian groups it is easy to see that a set of finite Haar measure is a determining set for finite complex measures if and only if