5. Pompeiu property in non-compact symmetric spaces

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.04.2024**

We now state two partial answers to the conjecture that seem to support the conjecture.

THEOREM 4.1 (Berenstein [3]). *Let* Ω *be a simply connected bounded open subset of* $\mathbf{R}^2$ *with* $C^{2+\varepsilon}$ *boundary, where* $\varepsilon > 0$. *Assume that the boundary value problem (4.1) has solutions for infinitely many positive* γ, *then* Ω *is a disc.*

We need some notation for the next result due to Brown and Kahane ([10]). Let Ω be a convex bounded open connected subset of $\mathbf{R}^2$. For $0 \leqslant \theta < \pi$, let $\omega(\theta)$ be the distance between the two parallel support lines for Ω which make an angle θ with the positive real axis. We assume ∂D is smooth so that ω is a continuous function. Let

$$m(\Omega) = \inf\{\omega(\theta): 0 \leqslant \theta < \pi\} \quad \text{and} \quad M(\Omega) = \sup\{\omega(\theta): 0 \leqslant \theta < \pi\}.$$

THEOREM 4.2 (Brown and Kahane [10]). *Let* Ω *be a convex region of* $\mathbf{R}^2$ *with* $\partial\Omega$ *real analytic. If* $m(\Omega) \leqslant \frac{1}{2} M(\Omega)$, *then* Ω *has the Pompeiu property.*

We remark that the proof of this Theorem is elementary and very elegant.

5. POMPEIU PROPERTY IN NON-COMPACT SYMMETRIC SPACES

Let G *be a connected non-compact semisimple Lie group having finite centre and real rank 1.* Let K be a fixed maximal compact subgroup of G. The space G/K is then a globally symmetric space of the non-compact type of rank 1. G/K is equipped with a natural Riemannian structure with respect to which G acts as a group of isometries and the associated Riemannian volume element μ is G-invariant. The basic results for the Pompeiu problem in this set-up are due to Berenstein and Zalcman ([9], [4]) and Berenstein and Shahshahani ([7]). In [9], the Fourier-analytic characterisation of a set — in fact, more generally, a collection of sets — having the pompeiu property is obtained and some explicit computations are made for geodesic spheres. In [7], the Pompeiu problem is reduced to an eigenvalue problem as in Section 4 and the analogue of Williams's results is obtained. We shall mainly present here a result implicit in the work of Berenstein and Zalcman as well as Berenstein and Shahshahani from our point of view of spectral analysis developed in [1]. It is to be noted that the close connection between the Pompeiu problem and spectral analysis is also developed in [4]. Our treatment, however, is different in that it relies on the results of spectral analysis in [1].

Let $G = KAN$ be the Iwasawa decomposition of G. Let $\mathcal{G}$ be the Lie algebra of G, $\mathcal{A}$ the Lie subalgebra of $\mathcal{G}$ corresponding to A. Since G has rank 1, $\mathcal{A}$ is 1-dimensional. Using the linear functional ρ on $\mathcal{A}$, which is the half-sum of the positive roots for the adjoint action of $\mathcal{A}$ on $\mathcal{G}$, we write the real dual $\mathcal{A}^*$ of $\mathcal{A}$ as $\mathcal{A}^* = \{t\rho, t \in \mathbf{R}\}$ and its complexification $\mathcal{A}_c^* = \{\lambda\rho, \lambda \in \mathbf{C}\}$. For $\lambda \in \mathbf{C}$, we denote by ϕ_λ the elementary spherical function associated with $\lambda\rho \in \mathcal{A}_c^*$. (These functions essentially parametrize the so-called class-1 representations of G.) The Weyl group in this case is the group of order 2 generated by the reflection $\lambda\rho \to -\lambda\rho$ and we have $\phi_{\lambda'} = \phi_\lambda$ if and only if $\lambda = \lambda'$ or $\lambda = -\lambda'$.

For $\lambda \in \mathbf{C}$ and k a non-negative integer, we write

$$\phi_{\lambda,k}(x) = d^k/d\lambda^k \, \phi_\lambda(x), \quad x \in G;$$

in particular, $\phi_{\lambda,0} = \phi_\lambda$. The functions $\{\phi_{\lambda,k}\}$ are K-bi-invariant, i.e.

$$\phi_{\lambda,k}(\kappa g \kappa') = \phi_{\lambda,k}(g), \quad \kappa \in K, \quad g \in G.$$

We denote by $\mathcal{E} = C^\infty(K\backslash G/K)$ the space of all K-bi-invariant C^∞-functions on G, with the topology of uniform convergence on compacta along with "derivatives". By $\mathcal{E}'$ we denote the dual space of $\mathcal{E}$, the space of K-bi-invariant distributions on G of compact support. A closed subspace $\mathcal{U} \subseteq \mathcal{E}$ is called a variety if $T * f \in \mathcal{U}$, whenever $T \in \mathcal{E}'$ and $f \in \mathcal{U}$ (here $*$ denotes convolution in G). The main theorem of [1] can be stated as follows:

THEOREM 5.1. *Let $\mathcal{U} \subseteq \mathcal{E} = C^\infty(K\backslash G/K)$ be a variety. Then $\mathcal{U}$ is the closed linear span in $\mathcal{E}$ of the subset $\{\phi_{\lambda,k} : \lambda \in \mathbf{C}, k \geqslant 0, \phi_{\lambda,k} \in \mathcal{U}\}$. In particular, if $\mathcal{U}$ is nonzero, then there exists $\lambda \in \mathbf{C}$ such that $\phi_\lambda \in \mathcal{U}$.*

We point out that the main ingredients of the proof of Theorem 5.1 are Schwartz's theorem on mean periodic functions on $\mathbf{R}$, and the topological isomorphism of $C_c^\infty(K\backslash G/K)$ with a space of entire functions through the spherical Fourier transform

$$f \to \hat{f}(\lambda) = \int_G f(x)\phi_\lambda(x^{-1})dx, \quad f \in C_c^\infty(K\backslash G/K).$$

The details of this topological isomorphism are also available in [4].

The spherical functions $\{\phi_\lambda : \lambda \in \mathbf{C}\}$ are intimately related to the *class-1 principal series representations* $\{\Pi_\lambda : \lambda \in \mathbf{C}\}$ of G. These representations are all realised in the space $L^2(K/M)$ (with normalised Haar measure dk; here, M is the centraliser of A in K). In fact, for $g \in G$ the operator $\Pi_\lambda(g)$ is:

$$\Pi_\lambda(g)(F)(k) = e^{(i\lambda-\rho)H(g^{-1}k)}F(\kappa(g^{-1}k)), \quad F \in L^2(K/M), \quad k \in K$$

where for any $y \in G$, $y = \kappa(y)\exp H(y)n(y)$ is the Iwasawa decomposition of y with exp denoting the exponential map: $\mathscr{A} \to A$. Then, $\phi_\lambda(g) = \langle \Pi_\lambda(g)1, 1\rangle$, $g \in G$, where 1 is the constant function in $L^2(K/M)$ and $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(K/M)$.

For a function f on $X = G/K$, we shall denote by $\tilde{f}$ the right K-invariant lift of f on G: $\tilde{f}(g) = f(gK)$. Similarly, if $E \subseteq X$, we write $\tilde{E} = \{g \in G : gK \in E\}$. As before, if f is a function on G, we denote by $\check{f}$ the function

$$\check{f}(g) = f(g^{-1}), \quad g \in G.$$

Note that $\int_G \tilde{f}(g)dg = \int_X f(x)d\mu(x)$ where μ is the volume element on G/K.

We are now in a position to state and prove the main theorem of this section which is implicit in the work of Berenstein and Zalcman ([9]) and Berenstein and Shahshahani ([7]), though not stated in this form.

THEOREM 5.2. *A relatively compact measurable subset* E *of* X *of positive measure has the Pompeiu property if and only if*

$$\Pi_\lambda(1_{\tilde{E}}) \overset{\text{def}}{=} \int_{\tilde{E}} \Pi_\lambda(g)\, dg$$

is a nonzero operator for every $\lambda \in \mathbf{C}$.

Proof. Given $E \subseteq X$ as in the statement, we notice that for $f \in C^\infty(X)$, $\int_{gE} f = 0$ for all $g \in G$ is equivalent to the condition $\tilde{f} * \check{1}_{\tilde{E}} = 0$. Define

$$\mathscr{U} = \{h \in C^\infty(G) : h(gk) = h(g) \quad \text{for all} \quad g \in G, k \in K, \quad \text{and} \quad h * \check{1}_{\tilde{E}} = 0\}.$$

Notice that $\mathscr{U}$ is a closed subspace invariant under left translation by elements of G. Then E has the Pompeiu property if and only if $\mathscr{U} = \{0\}$. Writing $\mathscr{V} = \mathscr{U} \cap \mathscr{E}$ (i.e., $\mathscr{V}$ is the space of K-bi-invariant functions in $\mathscr{U}$), we claim that $\mathscr{U} = \{0\}$ if and only if $\mathscr{V} = \{0\}$. First, if $\mathscr{U} \neq \{0\}$, choose $f \in \mathscr{U}$ such that $f(e) \neq 0$ where e is the identity element of G. Now for such an f, define the function h by

$$h(x) = \int_K f(kx)dk\,, \quad x \in G\,.$$

Since $\mathscr{U}$ is translation-invariant, $h \in \mathscr{U}$. On the other hand h is K-bi-invariant. So $h \in \mathscr{V} \subseteq \mathscr{U}$. But $h(e) = f(e) \neq 0$. It is now easy to show that $\mathscr{V}$ is a variety and by Theorem 5.1, $\mathscr{V} \neq \{0\}$ if and only if for some $\lambda \in \mathbf{C}$, $\phi_\lambda \in \mathscr{V}$.

Suppose now $\Pi_\lambda(1_{\tilde{E}}) = 0$ for some $\lambda \in \mathbf{C}$. We have for all $g \in G$,

$$\Pi_\lambda(g)\Pi_\lambda(1_{\tilde{E}}) = \int_{\tilde{E}} \Pi_\lambda(gx)dx = 0\,.$$

Consequently,

$$\begin{aligned} <\Pi_\lambda(g)\Pi_\lambda(1_{\tilde{E}})1, 1> &= \int_{\tilde{E}} <\Pi_\lambda(gx)1, 1> dx \\ &= \phi_\lambda * \check{1}_{\tilde{E}}(g) = 0\,. \end{aligned}$$

So $\phi_\lambda \in \mathscr{V}$ and hence E does not have the Pompeiu property. Conversely, suppose E does not have the Pompeiu property. Then $\mathscr{V} \neq \{0\}$ and hence there exists $\lambda \in \mathbf{C}$ such that $\phi_\lambda \in \mathscr{V}$. Further, if $\lambda \in i\mathbf{R}$, then $\phi_\lambda(x) > 0$ for all $x \in G$ and so $\phi_\lambda \notin \mathscr{V}$ as E has positive Haar measure. Thus $\lambda \notin i\mathbf{R}$, and then it is known that the representation Π_λ is irreducible (see [7]) and we have, for all $g \in G$,

$$\begin{aligned} \phi_\lambda * \check{1}_{\tilde{E}}(g) &= <\Pi_\lambda(g)\Pi_\lambda(1_{\tilde{E}})1, 1> \\ &= <\Pi_\lambda(1_{\tilde{E}})1, \Pi_\lambda^*(g)1> \\ &= <\Pi_\lambda(1_{\tilde{E}})1, \Pi_{\bar{\lambda}}(g^{-1})1> \\ &= 0\,. \end{aligned}$$

Since $\bar{\lambda} \notin i\mathbf{R}$, $\Pi_{\bar{\lambda}}$ is an irreducible representation; and hence 1 is a cyclic vector for $\Pi_{\bar{\lambda}}$. Our identity now implies $\Pi_\lambda(1_{\tilde{E}})1 = 0$. But since Π_λ is an irreducible class-1 representation and $1_{\tilde{E}}$ is a right K-invariant function, it follows that $\Pi_\lambda(1_{\tilde{E}})F = 0$ for all $F \in L^2(K/M)$. (Here we use the general fact that if Π_λ is a class-1 principal series representation and h is a right K-invariant function, then $\Pi_\lambda(h)$ is completely determined by its action on the constant function 1 on K/M. In fact, $\Pi_\lambda(f) = 0$ on the orthogonal complement of 1 in $L^2(K/M)$.) Thus $\Pi_\lambda(1_{\tilde{E}}) = 0$, proving the theorem.

As in Euclidean spaces, it is not possible to verify the condition of Theorem 5.2 in very many cases. Theorem 5.2 may also be taken as a starting point for considering the differential equations formulation of the

Pompeiu problem as in Berenstein and Shahshahani [7]. We quote a main result of theirs (their Proposition 3 and Corollary 1).

THEOREM 5.3 (Berenstein and Shahshahani). *Let* Ω *be an open relatively compact subset of* $X = G/K$ *such that* $X - \Omega$ *is connected and* $\partial\Omega$ *is Lipschitz. Assume that* Ω *does not have the Pompeiu property. Then* $\partial\Omega$ *is analytic.*

It follows from Theorem 5.3 that, for instance geodesic triangles in X have the Pompeiu property. On the other hand, Berenstein and Zalcman ([9]) as well as Berenstein and Shahshahani ([7]) point out that geodesic balls in X do not have the Pompeiu property. In fact, Sitaram ([29]) proves that if E is a relatively compact K-bi-invariant set and $m(E) > 0$, then E does not have the Pompeiu property. As in $\mathbf{R}^n$, it remains an open question whether the only sets with smooth boundary and connected exterior that fail to have the Pompeiu property are the geodesic balls. Analogues of Theorem 4.1 are also available for certain symmetric spaces (see [3] and [8]).

6. SYMMETRIC SPACES OF THE COMPACT TYPE

Let X be a symmetric space of the compact type, i.e., X is of the form G/K where G is a connected compact semisimple Lie group and K a suitable closed subgroup. Then (G, K) is a so-called Riemannian symmetric pair. G/K is equipped with a canonical G-invariant measure (see [15] for details).

As in the previous section, we shall connect the Pompeiu property with certain representations of the group G. We need only unitary irreducible representations Π of G on finite-dimensional Hilbert spaces $\mathscr{H}$. Such a representation Π is said to be a class-1 representation with respect to K if there exists a vector $0 \neq v \in \mathscr{H}$, such that for all $k \in K, \Pi(k)v = v$. For (G, K) a Riemannian symmetric pair it is well known that an irreducible representation Π is either of class-1 (in which case v is unique upto scalar multiples) or Π does not admit any nonzero vector v such that $\Pi(k)v = v$ for all $k \in K$ (see [16], p. 412). We record two simple observations for a continuous unitary irreducible representation Π of G: For $f \in L^1(G)$, we denote by $\Pi(f)$ the operator $\Pi(f) = \int_G \Pi(x) f(x)\, dx$. So if f is either right or left K-invariant, then $\Pi(f) = 0$ provided Π is not of class-1. If Π is class-1 and $0 \neq v_0$ is a K-fixed vector for Π and f is right K-invariant, then $\Pi(f) = 0$ on $(\mathbf{C} v_0)^{\perp}$, i.e., $\Pi(f)$ is completely determined by its action on v_0. On the other