

### 3. POMPEIU PROBLEM FOR THE $M(2)$ ACTION ON $\mathbb{R}^2$

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$$\begin{aligned}
&= \Sigma(S) * (\Sigma(T) * \tilde{\Sigma}(f))^\vee(0) \\
&\quad (\text{where } g^\vee(x) = g(-x), g \in \mathcal{E}(\mathbf{R}^2), x \in \mathbf{R}^2), \\
&= \Sigma(S) * ((\tilde{\Sigma}f)^\vee * \Sigma(T)^\vee)(0) \\
&= \Sigma(S) * (\tilde{\Sigma}(f) * \Sigma(T)) \text{ as } \Sigma(T), \tilde{\Sigma}(f) \text{ are even} \\
&= \Sigma(S) * (\Sigma(T) * \tilde{\Sigma}f)(0) \\
&= \langle \Sigma(S * T), \tilde{\Sigma}f \rangle \\
&\quad (\text{using } \Sigma S * \Sigma T = \Sigma S * T) \\
&= \langle S * T, f \rangle \\
&= S * T * f(0) \text{ as } f \text{ is even} \\
&= \langle S, T * f \rangle .
\end{aligned}$$

On the other hand,

$$\langle \Sigma(S), \tilde{\Sigma}(T * f) \rangle = \langle S, T * f \rangle .$$

The lemma is proved.

Finally, we come to the main result of the section: the spectral analysis theorem for radial functions. As we remarked in the introduction, the development in this section is along the same lines as in [1] where the corresponding result for rank-1 non-compact symmetric spaces is proved.

**THEOREM 2.4.** *Let  $\mathcal{V}$  be a closed nonzero subspace of  $\mathcal{E}(\mathbf{R}^2)_{\text{rad}}$  such that for all  $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$  and  $f \in \mathcal{V}$ ,  $T * f \in \mathcal{V}$ . Then there exists  $\lambda \in \mathbf{C}$  such that  $\phi_\lambda \in \mathcal{V}$ .*

*Proof.* Consider the closed and nontrivial subspace  $M$  of  $\mathcal{E}(\mathbf{R})_e$  such that  $\tilde{\Sigma}(\mathcal{V}) = M$ . By Lemma 2.3,  $M$  is closed under convolution with elements  $S \in \mathcal{E}'(\mathbf{R})_e$ . By the remarks following Theorem 2.1 now, there exists  $\lambda \in \mathbf{C}$  such that the functions  $\Psi_\lambda \in M$ , where  $\Psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$ ,  $x \in \mathbf{R}$ . A simple calculation now shows

$$\langle \phi_\lambda, f \rangle = \langle \Psi_\lambda, \Sigma f \rangle \quad f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}} \subseteq \mathcal{E}'(\mathbf{R}^2)_{\text{rad}} .$$

Thus  $\tilde{\Sigma}\phi_\lambda = \Psi_\lambda$  and hence  $\phi_\lambda \in \mathcal{V}$ .

### 3. POMPEIU PROBLEM FOR THE $M(2)$ ACTION ON $\mathbf{R}^2$

The Euclidean motion group  $M(2)$  is the semidirect product of  $\mathbf{R}^2$  with the rotation group  $SO(2, \mathbf{R})$ .

$$M(2) = \{(x, \sigma) : x \in \mathbf{R}^2, \sigma \in SO(2, \mathbf{R})\}$$

where

$$(x, \sigma) \cdot (x', \sigma') = (x + \sigma x', \sigma \sigma')$$

is the group multiplication and an element  $(x, \sigma)$  acts on  $y \in \mathbf{R}^2$  by the rule  $(x, \sigma)y = \sigma y + x$ .

Let  $E$  be a relatively compact subset of  $\mathbf{R}^2$  of positive Lebesgue measure. If  $f \in C(\mathbf{R}^2)$ , the space of continuous functions on  $\mathbf{R}^2$ , the vanishing of the integrals

$$\int_{gE} f(x) dx = 0, \text{ for all } g \in M(2)$$

i.e.  $\int_{\sigma E + y} f(x) dx = 0, \text{ for all } \sigma \in SO(2, \mathbf{R}), y \in \mathbf{R}^2$

can be restated as  $f * \check{1}_{\sigma E} \equiv 0$ , for all  $\sigma \in SO(2, \mathbf{R})$  or, equivalently  $f^\sigma * \check{1}_E \equiv 0$  for all  $\sigma \in SO(2, \mathbf{R})$ , where  $f^\sigma(x) = f(\sigma x)$  and  $\check{1}_E(x) = 1_E(-x)$ ,  $x \in \mathbf{R}^2$ . We write

$$\mathcal{U} = \{f \in \mathcal{E}(\mathbf{R}^2) : f^\sigma * \check{1}_E = 0 \text{ for all } \sigma \in SO(2, \mathbf{R})\}.$$

From elementary smoothing arguments, it follows that  $E$  has the Pompeiu property if and only if  $\mathcal{U} = \{0\}$ .  $\mathcal{U}$  is a closed subspace of  $\mathcal{E}(\mathbf{R}^2)$  which is invariant under translation and rotation. Let again

$$\mathcal{V} = \{f \in \mathcal{E}(\mathbf{R}^2)_{\text{rad}} : f * \check{1}_E = 0\}.$$

Then  $\mathcal{V} \subseteq \mathcal{U}$ ,  $\mathcal{V}$  is a closed subspace of  $\mathcal{E}(\mathbf{R}^2)_{\text{rad}}$  and  $T * \mathcal{V} \subseteq \mathcal{V}$  for all  $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$ .

We now prove the main theorem of [12] mentioned in the Introduction. (However, we restrict ourselves to indicator functions of sets, rather than general distributions of compact support.)

**THEOREM 3.1** (Brown, Schreiber and Taylor). *A relatively compact subset  $E \subseteq \mathbf{R}^2$  of positive Lebesgue measure does not have the Pompeiu property if and only if there exists  $\alpha \in \mathbf{C}, \alpha \neq 0$  such that*

$$\hat{1}_E(z_1, z_2) = 0 \text{ whenever } z_1^2 + z_2^2 = \alpha^2,$$

where  $\hat{1}_E$  is the Laplace-Fourier transform of the characteristic function  $1_E$  of  $E$ .

*Proof.* The *if* part is immediate; for instance, take any  $z = (z_1, z_2)$  such that  $z_1^2 + z_2^2 = \alpha^2$  and consider the function  $e^{iz \cdot x}$ . To prove the *only if* part, suppose  $E$  has the Pompeiu property. Let  $\mathcal{U}$  and  $\mathcal{V}$  be defined as above; by assumption we have  $\mathcal{U} \neq \{0\}$ . We shall now prove that  $\mathcal{V} \neq \{0\}$ . Choose  $f \in \mathcal{U}$  with  $f(0) \neq 0$  (this is possible as  $\mathcal{U}$  is translation-invariant). Define

$$h(y) = \int_{SO(2, \mathbf{R})} f(\sigma y) d\sigma, \quad y \in \mathbf{R}^2.$$

As  $\mathcal{U}$  is  $SO(2, \mathbf{R})$ -invariant, the function  $h \in \mathcal{U}$ . Further,  $h$  is a radial function by definition, so  $h \in \mathcal{V}$ . But then  $h(0) = f(0) \neq 0$ . Thus  $\mathcal{V} \neq \{0\}$  and by Theorem 2.2, we have  $\lambda \in \mathbf{C}$  such that  $\phi_\lambda \in \mathcal{V}$ . Further,  $\phi_0$  is the constant function and hence  $\phi_0$  cannot belong to  $\mathcal{V} \subseteq \mathcal{U}$ . So  $\lambda \neq 0$  and  $\phi_\lambda \in \mathcal{V}$  and, in particular,  $\phi_\lambda * \hat{1}_E(0) = 0$ . In the notation of Section 2, this means  $\mathcal{G}1_E(\lambda) = 0$  and hence  $\hat{1}_E(\lambda, 0) = 0$ . The  $SO(2, \mathbf{R})$ -invariance of  $\mathcal{U}$  now shows that  $\hat{1}_E$  vanishes on  $SO(2, \mathbf{R}) \cdot (\lambda, 0)$ . The analyticity argument in Lemma 2.2 will now prove that  $\hat{1}_E$  vanishes at all  $(z_1, z_2)$  where  $z_1^2 + z_2^2 = \lambda^2$ . This proves the theorem.

The condition in Theorem 3.1 can also be given a representation theoretic interpretation in terms of the so-called class-1 principal series representation of  $M(2)$  — see Section 7.3 for a more precise statement. As we remarked earlier, the condition of the theorem is verifiable only for sets having strong geometric properties. We quote two results from [12] without proof.

**THEOREM 3.2** (Brown, Schreiber and Taylor). *The ellipse*

$$E = \left\{ (x, y) \in \mathbf{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

*has the Pompeiu property if and only if  $a, b > 0$  and  $a \neq b$ .*

When  $a = b > 0$ ,  $D$  is the disc and we have

$$\hat{1}_D(z_1, z_2) = \text{const. } J_1(\sqrt{|z_1^2 + z_2^2|}) / \sqrt{|z_1^2 + z_2^2|}, \quad (z_1, z_2) \in \mathbf{C}^2,$$

where  $J_1$  is the Bessel function. Since there are infinitely many zeros of  $J_1$ ,  $D$  does not have the Pompeiu property. The next theorem, obtained through a careful estimate ([12]) needs a definition.

**Definition 3.3.** Let  $\Gamma = \Gamma(t)$ ,  $-1 \leq t \leq 1$  be a Lipschitz curve in  $\mathbf{R}^2$  with well defined (a.e.) unit tangent vectors  $T(t) = \Gamma'(t) / |\Gamma'(t)|$ . The point  $p = \Gamma(0)$  is a corner of  $\Gamma$  if both the right and the left limits of  $T(t)$  as  $t \rightarrow 0$  exist and are not multiples of each other.

**THEOREM 3.4** (Brown, Schreiber and Taylor). *Let  $\Omega$  be a compact connected subset of  $\mathbf{R}^2$ . Suppose that there is a half-plane  $H$  and a unique point  $p \in \Omega \cap H$  of maximal distance from the boundary  $\partial H$  of  $H$ . If the boundary of  $\Omega$  near  $p$  is given by a Lipschitz curve with a corner at  $p$  then  $\Omega$  has the Pompeiu property.*

Let now  $\Omega$  be a bounded Borel subset of the plane of positive measure and suppose that  $\Omega$  does not have the Pompeiu property. By Theorem 3.1,  $\hat{1}_\Omega$  vanishes on the algebraic variety  $M_\alpha = \{(z_1, z_2): z_1^2 + z_2^2 = \alpha^2\}$  for some  $\alpha \neq 0$ . As observed in [33] and [34],  $\hat{1}_\Omega/(z_1^2 + z_2^2 - \alpha^2)$  is now an entire function and standard Paley-Wiener theorem yields the following proposition.

**PROPOSITION 3.5.** *If  $\Omega$  is a bounded Borel subset of  $\mathbf{R}^2$  of positive measure with  $\hat{1}_\Omega$  vanishing on  $M_\alpha, \alpha \neq 0$ , then the function  $g(z_1, z_2) = \hat{1}_\Omega/(z_1^2 + z_2^2 - \alpha^2)$  is an entire function on  $\mathbf{C}^2$  which is the Laplace-Fourier transform of a distribution of compact support.*

Proposition 3.5 immediately gives rise to a partial differential equation. For, if  $T$  is the distribution whose Fourier transform is  $g$ , then from

$$(z_1^2 + z_2^2 - \alpha^2)g(z_1, z_2) = \hat{1}_\Omega(z_1, z_2)$$

we have

$$(3.1) \quad \Delta T + \alpha^2 T = -1_\Omega$$

where  $\Delta$  is the Laplacian. Conversely, if there exists a distribution  $T$  of compact support satisfying the equation (3.1), then  $\hat{1}_\Omega$  vanishes on  $M_\alpha$  and hence  $\Omega$  does not have the Pompeiu property. We also remark that if  $\Omega$  is, further, a bounded simply connected open set and the equation (3.1) has a solution, then  $\alpha^2$  is necessarily a positive real number as can be seen from a simple Green's theorem argument (see [34] for a proof). The equation has been studied in [3], [33] and [34]. In [33] it was proved that a solution of (3.1), if it exists is actually a function. We shall discuss some more of these results in the next section. We end the present section by quoting the main theorem of [34]. This result extends Theorem 3.4 and, barring sets of rotational symmetry all known sets failing to have the Pompeiu property are covered by this result. For a bounded subset  $\Omega \subseteq \mathbf{R}^2$ , we denote by  $\partial^*\Omega$  the boundary of the unbounded component.

THEOREM 3.5 (S. A. Williams). *Let  $\Omega$  be a bounded open subset such that the equation  $\Delta T + \alpha^2 T = -1_\Omega$  has a function solution of compact support for some  $\alpha > 0$ . Let  $R, K, L$  be positive real numbers such that  $L > KR$ . Assume that for  $P \in \partial^* \Omega$  there exists a coordinate system  $(x, y)$  around  $P$  so that*

(i)  $Q = (-R, R) \times (-L, L)$  intersects  $\partial\Omega$  in the graph  $y = f(x)$  of a Lipschitz function  $f$  with Lipschitz constant  $K$ , and

(ii)  $Q \cap \Omega = \{(x, y) : |x| < R \text{ and } f(x) < y < L\}$ .

Then  $f$  is real analytic in a neighbourhood of  $P$ .

Thus if we restrict ourselves to the class  $\mathcal{D}$  of simply connected bounded open sets with Lipschitz boundary then  $\Omega \in \mathcal{D}$  can fail to have the Pompeiu property only if  $\partial\Omega$  is real analytic.

#### 4. A LONG-STANDING CONJECTURE !

The following Conjecture has received quite some attention in the literature ([3], [10], [34]).

*Conjecture.* If  $\Omega \subseteq \mathbf{R}^2$  is in the class  $\mathcal{D}$  described above and if  $\Omega$  does not have the Pompeiu property, then  $\Omega$  is a disc.

As pointed out before, the work of Williams shows that it is enough to consider  $\Omega$  with  $\partial\Omega$  real analytic. For  $\Omega \in \mathcal{D}$ , the existence of (a necessarily positive)  $\alpha^2$  for which (3.1) has a distribution solution of compact support is equivalent to the existence of a positive  $\gamma$  for which the following overdetermined system has a solution.

$$(4.1) \quad \Delta T + \gamma T = 0 \quad \text{on } \Omega$$

$$T = \text{constant} \neq 0 \quad \text{on } \partial\Omega, \quad \partial T / \partial n \equiv 0 \quad \text{on } \partial\Omega$$

(see [34] for details). Thus the conjecture can be stated as follows:

If for  $\Omega \in \mathcal{D}$ , there exists  $\gamma > 0$  for which (4.1) admits a solution, then  $\Omega$  is a disc.

It is remarked in [34] that the conjecture is closely related to a result of Serrin ([25]): If  $\Omega$  is a bounded connected open set with smooth boundary on which

$$\Delta u = -1 \quad \text{on } \Omega$$

$$u = 0, \quad \partial u / \partial n = \text{constant} \quad \text{on } \partial\Omega$$

has a function solution, then  $\Omega$  must be a disc.