V. Graph polynomials

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where $<K \mid L>$ denotes the product of vertex weights (all relative to the given template). Independence of the template follows from the well-definedness of the polynomial itself.

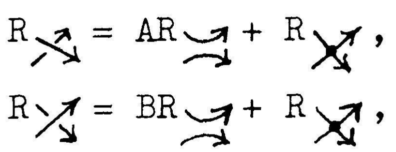
Remark. It would be very interesting to know the relationship between this state model for the Kauffman polynomial and the extraordinary model of Jaeger [34]. Jaeger gives a state expansion where the states are a collection of oriented knots and links. Each state is itself evaluated via the regular isotopy version of the Homfly polynomial.

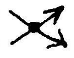
V. Graph polynomials

The two skein polynomials (Homfly and Kauffman) each have three variable extensions to rigid vertex isotopy invariants of 4-valent graphs imbedded in three-space. This construction has been announced in [45]. (See also [56] and [74].) Our skein models involve 4-valent graphs implicitly, and so give rise to a natural definition for these extended polynomials as state models.

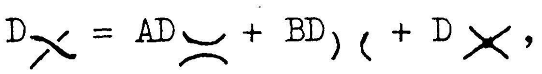
Let the new variables A and B be given, with $z = A - B$ the usual z for the skein polynomials. The extended polynomials are then defined by the axioms:

Homfly Extension Axioms

1. 

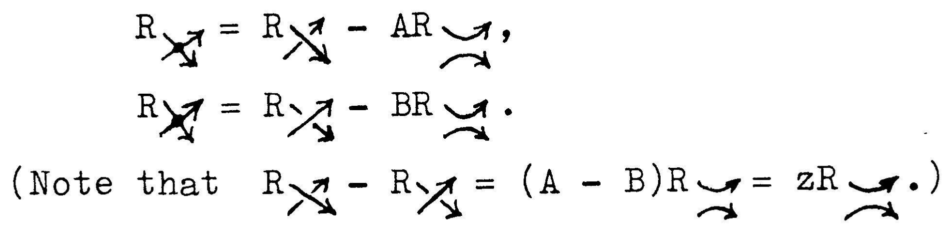
2. R_K = usual regular isotopy
   Homfly polynomial if K is
   free of graphical vertices ().

Kauffman Extension Axioms

1. 

2. D_K = usual regular isotopy
   Dubrovnik polynomial if K is
   free of graphical vertices ().

In each case, these axioms are taken as the *definition* of the polynomial in the case of the presence of graph vertices. That is, one rewrites



as the (recursive) definition in terms of an expansion involving the standard link polynomial (evaluated on a collection of links obtained by removing graph vertices and replacing by splices or by crossings). It is easy to verify that the resulting graph polynomial is well-defined *on the basis of the theory of the Homfly or Kauffman polynomial.*

By using the skein models we can give a direct formula for the evaluation of the extended polynomial on a planar graph. (See below.) This gives another point of view for these polynomials. In this view the axioms for the extended polynomials give state expansions for them whose states are plane graphs. Each plane graph state contributes a polynomial to the summation — weighted by A's and B's that tell how it was made planar by projecting and splicing.

In this view, all the complexity of the skein models for these polynomials has been absorbed into the extended polynomial evaluations for plane graphs.

Here are the explicit formulas for the Homfly case. I leave the case of the Kauffman extension to the reader. (See also [56].)

LEMMA 5.1. *Let G be a plane oriented 4-valent graph. Let $U(G)$ be the set of graphs obtained from G by splicing (oriented splice) a subset of vertices from G. Let T be a template for G (regard G as a universe and use the notion of template in section 2).*

For each H in $U(G)$ give H the structure of a standard unlink $L = L(H)$ relative to the template T. Let $t(H), t_+(H), t_-(H)$, denote the number of splices, positive splices, negative splices of $L(H)$. Let $w(H)$ denote the writhe of $L(H)$. Then

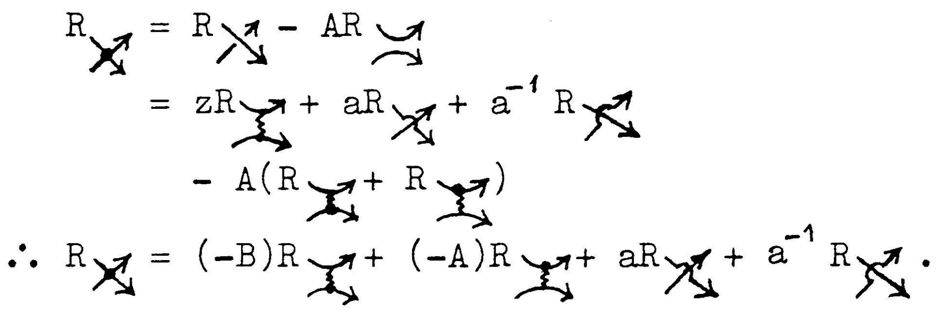
$$R_G = \sum_{H \in U(G)} (-1)^{t(H)} A^{t_-(H)} B^{t_+(H)} a^{w(H)} \delta^{|H|-1}$$

where

$$\delta = (a - a^{-1})/z = (a - a^{-1})/(A - B),$$

and $|H|$ denotes the number of crossing circuits in H (i.e. the number of link components in $L(H)$).

Proof. We give a state expansion for the extended polynomial by using the axiom. In abbreviated form this gives



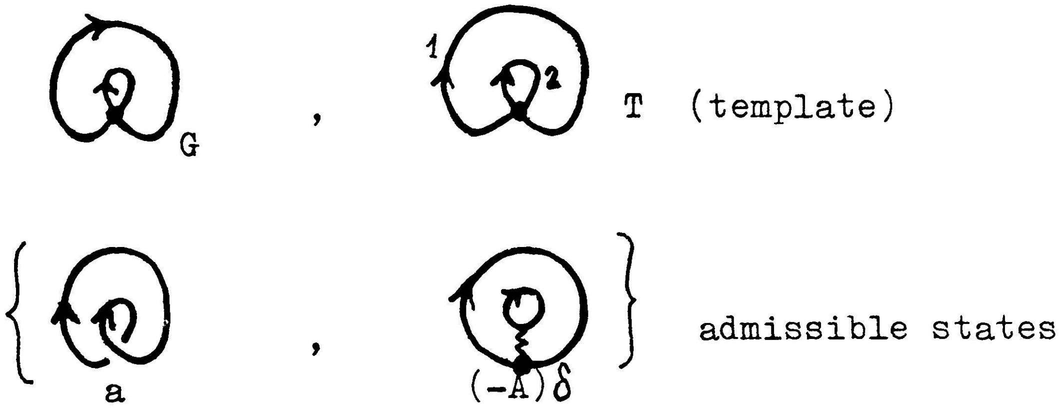
(z = A - B).

Hence, the vertex weights for R_G are $(-B)$ for a positive splice, $(-A)$ for a negative splice, and otherwise the same as in the model for the Homfly. This completes the proof.

In the case of both D_G and R_G for plane graphs it would be good to verify their properties and definedness directly and independently of the knot theory. The knot theory of the (extended) Homfly and Kauffman polynomials would then be seen to rest on this theory of plane graph polynomials.

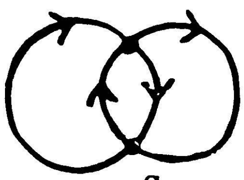
An example.

1.

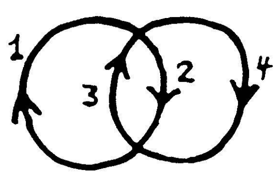


$$\therefore R_G = a - \frac{A(a-a^{-1})}{A-B} = \frac{Aa^{-1} - Ba}{A-B}$$

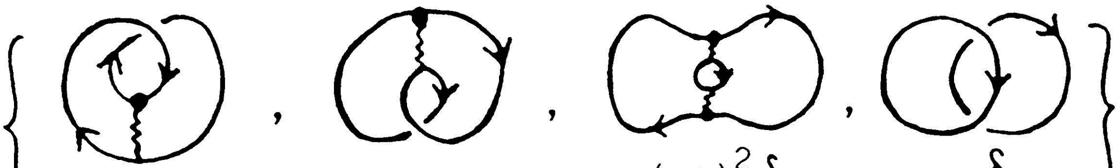
2.



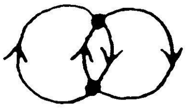
G



T (template)

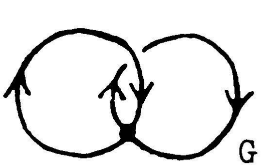


−Ba, −Aa, $(-A)^2\delta$, δ

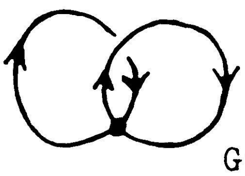
$\therefore R$  $= -Ba - Aa + (A^2+1)\dfrac{(a-a^{-1})}{A-B}$

$= [(a-a^{-1})+(B^2a-A^2a^{-1})]\,(A-B)$.

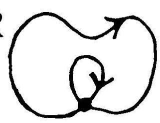
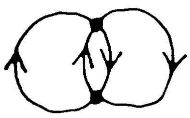
3.



G



G*

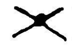
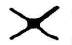
$R_G = AR$  $+ R$ 

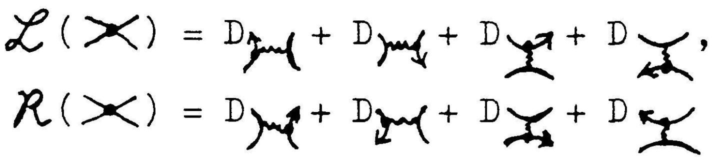
$$= A\left[\frac{Aa^{-1}-Ba}{A-B}\right] + \left[\frac{(a-a^{-1})+(B^2a-A^2a^{-1})}{A-B}\right]$$

$$= \frac{(a-a^{-1}) + (B^2-AB)a}{A-B}.$$

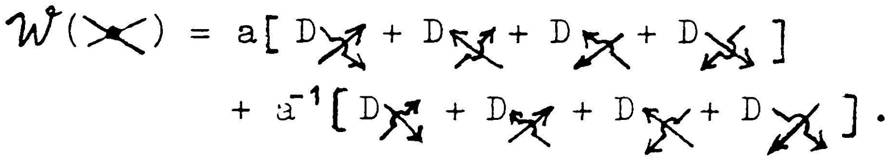
Here the rational function is not invariant under the (simultaneous) substitution of a by a^{-1} and interchange of A and B. This reflects the fact that the graph embeddings G and G^* are not rigid-vertex isotopic.

It is worth mentioning the planar graph polynomial in the Dubrovnik case. The result is

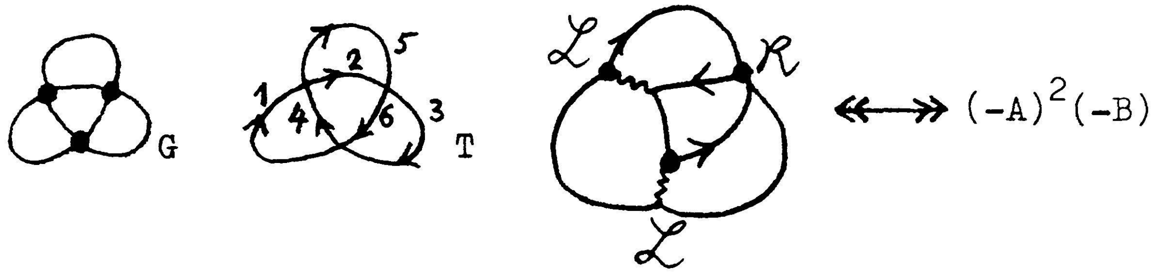
D  = (−A)$\mathcal{L}$() + (−B)$\mathcal{R}$() + $\mathcal{W}$(), where



and



A state in this expansion is obtained by first splitting (in any way) the vertices of the given unoriented four-valent plane graph G. The vertex weights are then determined by the template, as illustrated below.



If $<G \mid S>$ denotes the product of vertex weights for a given state S, then the polynomial has the form

$$D_G = \sum_S <G \mid S> \mu^{|S|-1}, \quad \mu = 1 + (a - a^{-1})/(A - B).$$

Proof of these formulas from the extension axioms follows just as in the Homfly case.

VI. The Conway Polynomial

The skein models give a very elegant formulation of the Conway polynomial ([16], [41]) (compare [33])

$$\nabla_K(z) = R_K(z, 1).$$

Specializing the formula for the skein model we have

$$\nabla_K(z) = \sum (-1)^{t_-(L)} z^{t(L)}$$