8. GAUSS'S REDUCTION PROCESS

(ii) When $f = p$ (prime) and p does not divide a, we set $I_1 = I$. If p divides a, we take for I the ideal $a_1[1, \phi_1]$ following I in its period. In this case, as $p \mid a$, from $p^2D = b_1^2 + 4aa_1$, we see that $p \mid b_1$ and so, as $GCD(a_1, b_1, a) = 1$ we see that p does not divide a_1. Then, by (2.12), we have $I_1 = \rho I$ with $\rho = \frac{a_1}{a}\phi_1$. Now, by Proposition 5, $\phi_1 = \frac{b_1 + \sqrt{D'}}{2a_1}$ is reduced, so that $1 \leqslant b_1 < \sqrt{D'}$, and

$$1 \leqslant a_1 < \sqrt{D'}\,, \tag{7.5}$$

giving

$$1 \leqslant \rho < \sqrt{D'}\,. \tag{7.6}$$

The rest of the proof follows exactly as in the proof of (i) using (7.5) (resp. (7.6)) in place of (7.3) (resp. (7.4)).

8. Gauss's reduction process

Definition 14. (Half-reduced) A representation $\{a, b\}$ of an ideal I is said to be *half-reduced* if

$$0 < \frac{-b + \sqrt{D}}{2\,|\,c\,|} < 1\,, \tag{8.1}$$

where $c = (D - b^2) \mid 4a$.

An ideal I is called *half-reduced* if there exists a half-reduced representation of I.

Clearly, if $\{a, b\}$ is half-reduced, then $b < \sqrt{D}$ and $\{-a, b\}$ is half-reduced.

LEMMA 7. *Let I be a primitive ideal of O_D. To each representation $\{a, b\}$ of I corresponds a unique integer q such that the q-neighbour representation $\{a', b'\}$ is half-reduced. The integer b' and the ideal $I' = \left[a', \frac{b' + \sqrt{D}}{2}\right]$ are determined by I. The value of q is*

$$q = \frac{a}{|\,a\,|}\left[\frac{b + \sqrt{D}}{2\,|\,a\,|}\right]. \tag{8.2}$$

The representation $\{a', b'\}$ *and the ideal* I' *are the Gauss neighbour of the representation* $\{a, b\}$ *and of the ideal* I *respectively, so that*

$$\{a, b\} \overset{G}{\to} \{a', b'\} .$$

Proof. As $c' = \dfrac{(D - b'^2)}{4a'} = a$ (by (2.10)), the q-neighbour representation $\{a', b'\}$ of $\{a, b\}$ is half-reduced if

$$0 < \frac{-b' + \sqrt{D}}{2\,|\,a\,|} < 1 ,$$

that is, by (2.10), if $0 < \dfrac{b + \sqrt{D}}{2\,|\,a\,|} - \dfrac{a}{|\,a\,|}\, q < 1$, giving $q = \dfrac{a}{|\,a\,|}\left[\dfrac{b + \sqrt{D}}{2|\,a\,|}\right]$, which shows that q and $\{a', b'\}$ are determined by $\{a, b\}$. Let $\{\pm a, b + 2K\,|\,a\,|\} = \{a_1, b_1\}$ be another representation of I giving rise to a half-reduced representation, say $\{a_1', b_1'\}$. As $b_1' \equiv -b_1 \equiv -b \equiv b' \pmod{2\,|a|}$ and $|\,a_1\,| = |\,a\,|$, we see from the inequalities

$$0 < \frac{\sqrt{D} - b'}{2\,|\,a\,|} < 1 \quad \text{and} \quad 0 < \frac{\sqrt{D} - b_1'}{2\,|\,a_1|} < 1$$

that $b_1' = b'$. Hence, as $|\,a\,| = |a_1|$ and $b' = b_1'$, from $D = b'^2 + 4aa' = b_1'^2 + 4a_1 a_1'$, we see that $|\,a'| = |\,a_1'\,|$. This shows that $I_1' = I$, which completes the proof of Lemma 7.

PROPOSITION 11. *Let* $\{a, b\}$ *be a half-reduced representation of a half-reduced ideal* I. *Let* $\{a, b\} \overset{G}{\to} \{a', b'\}$ *and set* $I' = \left[a', \dfrac{b' + \sqrt{D}}{2}\right]$. *We have*

(i) *if* $b < -\sqrt{D}$ *then* $b' > b + 2\sqrt{D}$,

(ii) *if* $b > -\sqrt{D}$ *then* I' *is reduced.*

(iii) *if* I *is reduced, then* I' *is reduced, and moreover if* $\{a, b\}$ *is the representation of* I *such that* $a > 0$ *and* $\phi = \dfrac{b + \sqrt{D}}{2a}$ *is reduced, then the Lagrange neighbour and the Gauss neighbour are the same.*

Proof. For any representation $\{a, b\}$ of any primitive ideal, we have

$$\left|\frac{\sqrt{D} - b}{2c}\right| \left|\frac{\sqrt{D} + b}{2a}\right| = 1 . \tag{8.3}$$

Now take $\{a, b\}$ to be a half-reduced representation of the half-reduced ideal I so that $0 < \dfrac{-b+\sqrt{D}}{2|c|} < 1$, where $c = (D-b^2)/4a$.

(i) Suppose that $b < -\sqrt{D}$. Then we have $b^2 - D = 4|a||c|$ so that (8.3) becomes $\left(\dfrac{\sqrt{D}-b}{2|c|}\right)\left(\dfrac{-b-\sqrt{D}}{2|a|}\right) = 1$. As $0 < \dfrac{-b+\sqrt{D}}{2|c|} < 1$, we see that $\dfrac{-b-\sqrt{D}}{2|a|} > 1$. But, as $\{a', b'\}$ is also half-reduced, we have $\dfrac{-b'+\sqrt{D}}{2|a|} < 1$, so that $-b'+\sqrt{D} < 2|a| < -b-\sqrt{D}$, proving that $b' > b + 2\sqrt{D}$.

(ii) Suppose that $b > -\sqrt{D}$. Then, we have $|b| < \sqrt{D}$, and (8.3) can be written

$$\left(\frac{\sqrt{D}-b}{2|c|}\right)\left(\frac{\sqrt{D}+b}{2|a|}\right) = 1 .$$

showing that $\dfrac{\sqrt{D}+b}{2|a|} > 1$. Or the other hand, as $\{a', b'\}$ is half-reduced, we have $0 < \dfrac{\sqrt{D}-b'}{2|a|} < 1$, that is $0 < \dfrac{\sqrt{D}+b}{2|a|} - \dfrac{a}{|a|}q < 1$, so that

$$\frac{a}{|a|}q = \left[\frac{\sqrt{D}+b}{2|a|}\right] \geqslant 1 .$$

Hence we obtain

$$\sqrt{D} + b' = \sqrt{D} - b + 2aq = (\sqrt{D}-b) + 2|a|\left(\frac{aq}{|a|}\right) > 2|a|,$$

which, together with the inequalities $0 < \dfrac{\sqrt{D}-b'}{2|a|} < 1$, shows that ϕ' is reduced if $a > 0$ and $-\phi'$ is reduced if $a < 0$, proving that I' is reduced.

(iii) We suppose that I is reduced and choose the representation $\{a, b\}$ of I with $a > 0$ and $\phi = \dfrac{b+\sqrt{D}}{2a}$ reduced. As ϕ is half-reduced and $b > -\sqrt{D}$ from (ii) we see that I' is reduced. Moreover, the integer q used to obtain both the Lagrange neighbour and the Gauss neighbour of $\{a, b\}$ is $[\phi]$. This shows that the two neighbours of $\{a, b\}$ are the same and concludes the proof of Proposition 11.

Definition 15. *(Gauss's reduction process* ([1]: §§ 183-185)) We start with a primitive ideal I_0 of O_D and a representation $\{a, b\}$ of I_0, and define the sequence of representations $\{a_n, b_n\}$ of the primitive ideals I_n by

$$\{a_n, b_n\} \overset{G}{\rightarrow} \{a_{n+1}, b_{n+1}\} \qquad (n = 0, 1, 2, \ldots) .$$

We now show that Gauss's reduction process leads to a reduced ideal equivalent to I_0. In addition we give an upper bound for the number of steps required to obtain a reduced ideal I_n as well as bounds for a quantity ρ in the relation $I_n = \rho I_0$.

PROPOSITION 12. *(i) The ideal I_n is reduced for*

$$n > \max\left(\frac{|a_0|}{\sqrt{D}} + 1, 2\right) .$$

(ii) Let I' be the first reduced ideal obtained by applying Gauss's reduction to I_0. Then $I = \rho I_0$ with $\dfrac{1}{|a_0|} \leqslant \rho < \sqrt{D}$.

Proof. We suppose that $n > \max\left(\dfrac{|a_0|}{\sqrt{D}} + 1, 2\right)$ so that $n \geqslant 3$.

If $b_1 > -\sqrt{D}$, by Proposition 11 (ii), I_2 is reduced and so, by Proposition 11 (iii), I_n is reduced.

Suppose on the other hand that $b_1 < -\sqrt{D}$ and that I_n is not reduced. Then, by Proposition 11 (ii), we see that $b_i < -\sqrt{D}$ for $i = 1, 2, \ldots, n-1$. Then, by Proposition 11 (i), we have

$$b_{n-1} > b_1 + 2(n-2)\sqrt{D} .$$

Hence we obtain

$$\begin{aligned} b_{n-1} &> -b_0 + 2a_0\left(\frac{a_0}{|a_0|}\left[\frac{a_0}{|a_0|}\,\frac{(b_0+\sqrt{D}}{2a_0}\right]\right) + 2\left(\frac{|a_0|}{\sqrt{D}} - 1\right)\sqrt{D} \\ &> -b_0 + 2|a_0|\left(\frac{b_0+\sqrt{D}}{2|a_0|} - 1\right) + 2\left(\frac{|a_0|}{\sqrt{D}} - 1\right)\sqrt{D} \\ &= -\sqrt{D} , \end{aligned}$$

which is a contradiction. This completes the proof that I_n is reduced for $n > \max\left(\dfrac{|a_0|}{\sqrt{D}} + 1, 2\right)$.

(ii) Let I_n be the first reduced ideal obtained from I_0 by Gauss's reduction process. If $n = 0$ then $\rho = 1$, so that $\frac{1}{|a_0|} \leqslant \rho < \sqrt{D}$. If $n \geqslant 1$ we have $I_n = \rho I_0$ with (by (2.12))

$$\rho = \left| \frac{a_1}{a_0} \phi_1 \ldots \frac{a_n}{a_{n-1}} \phi_n \right| = \left| \frac{a_n}{a_0} \right| \left| \frac{b_1 + \sqrt{D}}{2a_1} \right| \ldots \left| \frac{b_n + \sqrt{D}}{2a_n} \right| .$$

As the representations $\{a_k, b_k\}$ are half-reduced for $k \geqslant 1$, we see, by (8.3), that $\left| \frac{b_k + \sqrt{D}}{2a_k} \right| > 1\,(k \geqslant 1)$ so that $\rho > \left| \frac{a_n}{a_0} \right| \geqslant \frac{1}{|a_0|}$. On the other hand we have

$$\rho = \left| \frac{b_1 + \sqrt{D}}{2a_0} \right| \ldots \left| \frac{b_n + \sqrt{D}}{2a_{n-1}} \right| .$$

As $\{a_k, b_k\}$ is a half-reduced representation for $k = 1, 2, \ldots, n$, we have $0 < \sqrt{D} - b_k < 2\,|\,a_{k-1}\,|$. Furthermore, for $k = 1, 2, \ldots, n-1$, we have $\sqrt{D} + b_k < 2\,|\,a_{k-1}\,|$, as otherwise $0 < \sqrt{D} - b_k < 2\,|\,a_{k-1}\,| < \sqrt{D} + b_k$, which is equivalent to $0 < \sqrt{D} - b_k < 2\,|\,a_k\,| < \sqrt{D} + b_k$ so that by (4.2) the primitive ideal I_k would be reduced. Therefore, for $k = 1, 2, \ldots, n-1$, we have

$$|\sqrt{D} + b_k| \leqslant \sqrt{D} + |\,b_k\,| = \begin{cases} \sqrt{D} + b_k < 2\,|\,a_{k-1}\,|, & \text{if} \quad b_k \geqslant 0, \\ \sqrt{D} - b_k < 2\,|\,a_{k-1}\,|, & \text{if} \quad b_k < 0, \end{cases}$$

so that, as $\{a_n, b_n\}$ is reduced,

$$\rho < \frac{b_n + \sqrt{D}}{2\,|\,a_{n-1}\,|} < \sqrt{D}$$

which completes the proof of Proposition 12.

We remark that Proposition 7 and 12 suggest that Lagrange's reduction process may lead to a reduced ideal much faster than Gauss's reduction process, as the number M_0 of Lemma 6 is much smaller than $\max\left(\frac{|\,a_0\,|}{\sqrt{D}} + 1, 2\right)$.

Example 5. We apply both Lagrange reduction and Gauss reduction to the representation $\{3655, 7068\}$ of the primitive ideal $[3655, 3534 + \sqrt{21}]$ of O_{84}. We obtain

$$\{3655, 7068\} \xrightarrow{L} \{-3417, -7068\} \xrightarrow{L} \{4, 234\} \xrightarrow{L} \{3, 6\} \quad \text{(3 steps)}$$

and

$$\{3655, 7068\} \xrightarrow{G} \{-3417, -7068\} \xrightarrow{G} \{3187, -6600\} \xrightarrow{G} \{-2965, -6148\} \xrightarrow{G} \ldots$$
$$\xrightarrow{G} \{-1, -12\} \xrightarrow{G} \{-5, 8\} \quad \text{(30 steps)} .$$

We remark that M_0 is approximately 8.72 and $\dfrac{|a_0|}{\sqrt{D}} + 1$ is approximately 399.8.

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