## 8. GAUSS'S REDUCTION PROCESS

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(ii) When $f=p$ (prime) and $p$ does not divide $a$, we set $I_{1}=I$. If $p$ divides $a$, we take for $I$ the ideal $a_{1}\left[1, \phi_{1}\right]$ following $I$ in its period. In this case, as $p \mid a$, from $p^{2} D=b_{1}^{2}+4 a a_{1}$, we see that $p \mid b_{1}$ and so, as $\operatorname{GCD}\left(a_{1}, b_{1}, a\right)=1$ we see that $p$ does not divide $a_{1}$. Then, by (2.12), we have $I_{1}=\rho I$ with $\rho=\frac{a_{1}}{a} \phi_{1}$. Now, by Proposition $5, \phi_{1}=\frac{b_{1}+\sqrt{D^{\prime}}}{2 a_{1}}$ is reduced, so that $1 \leqslant b_{1}<\sqrt{D^{\prime}}$, and

$$
\begin{equation*}
1 \leqslant a_{1}<\sqrt{D^{\prime}}, \tag{7.5}
\end{equation*}
$$

giving

$$
\begin{equation*}
1 \leqslant \rho<\sqrt{D^{\prime}} \tag{7.6}
\end{equation*}
$$

The rest of the proof follows exactly as in the proof of (i) using (7.5) (resp. (7.6)) in place of (7.3) (resp. (7.4)).

## 8. GAUSS'S REDUCTION PROCESS

Definition 14. (Half-reduced) A representation $\{a, b\}$ of an ideal $I$ is said to be half-reduced if

$$
\begin{equation*}
0<\frac{-b+\sqrt{D}}{2|c|}<1 \tag{8.1}
\end{equation*}
$$

where $c=\left(D-b^{2}\right) \mid 4 a$.
An ideal $I$ is called half-reduced if there exists a half-reduced representation of $I$.

Clearly, if $\{a, b\}$ is half-reduced, then $b<\sqrt{D}$ and $\{-a, b\}$ is half-reduced.

Lemma 7. Let $I$ be a primitive ideal of $O_{D}$. To each representation $\{a, b\}$ of $I$ corresponds a unique integer $q$ such that the $q$-neighbour representation $\left\{a^{\prime}, b^{\prime}\right\}$ is half-reduced. The integer $b^{\prime}$ and the ideal $I^{\prime}=\left[a^{\prime}, \frac{b^{\prime}+\sqrt{D}}{2}\right]$ are determined by $I$. The value of $q$ is

$$
\begin{equation*}
q=\frac{a}{|a|}\left[\frac{b+\sqrt{D}}{2|a|}\right] . \tag{8.2}
\end{equation*}
$$

The representation $\left\{a^{\prime}, b^{\prime}\right\}$ and the ideal $I^{\prime}$ are the Gauss neighbour of the representation $\{a, b\}$ and of the ideal $I$ respectively, so that

$$
\{a, b\} \xrightarrow{G}\left\{a^{\prime}, b^{\prime}\right\} .
$$

Proof. As $c^{\prime}=\frac{\left(D-b^{\prime 2}\right)}{4 a^{\prime}}=a$ (by (2.10)), the $q$-neighbour representation $\left\{a^{\prime}, b^{\prime}\right\}$ of $\{a, b\}$ is half-reduced if

$$
0<\frac{-b^{\prime}+\sqrt{D}}{2|a|}<1
$$

that is, by (2.10), if $0<\frac{b+\sqrt{D}}{2|a|}-\frac{a}{|a|} q<1$, giving $q=\frac{a}{|a|}\left[\frac{b+\sqrt{D}}{2|a|}\right]$, which shows that $q$ and $\left\{a^{\prime}, b^{\prime}\right\}$ are determined by $\{a, b\}$. Let $\{ \pm a, b+2 K|a|\}=\left\{a_{1}, b_{1}\right\}$ be another representation of $I$ giving rise to a half-reduced representation, say $\left\{a_{1}^{\prime}, b_{1}^{\prime}\right\}$. As $b_{1}^{\prime} \equiv-b_{1} \equiv-b \equiv b^{\prime}(\bmod 2|a|)$ and $\left|a_{1}\right|=|a|$, we see from the inequalities

$$
0<\frac{\sqrt{D}-b^{\prime}}{2|a|}<1 \quad \text { and } \quad 0<\frac{\sqrt{D}-b_{1}^{\prime}}{2\left|a_{1}\right|}<1
$$

that $b_{1}^{\prime}=b^{\prime}$. Hence, as $|a|=\left|a_{1}\right|$ and $b^{\prime}=b_{1}^{\prime}$, from $D=b^{\prime 2}+4 a a^{\prime}$ $=b_{1}^{\prime 2}+4 a_{1} a_{1}^{\prime}$, we see that $\left|a^{\prime}\right|=\left|a_{1}^{\prime}\right|$. This shows that $I_{1}^{\prime}=I$, which completes the proof of Lemma 7 .

Proposition 11. Let $\{a, b\}$ be a half-reduced representation of a halfreduced ideal $I$. Let $\{a, b\} \xrightarrow{G}\left\{a^{\prime}, b^{\prime}\right\}$ and set $I^{\prime}=\left[a^{\prime}, \frac{b^{\prime}+\sqrt{D}}{2}\right]$. We have
(i) if $b<-\sqrt{\bar{D}}$ then $b^{\prime}>b+2 \sqrt{D}$,
(ii) if $b>-\sqrt{D}$ then $I^{\prime}$ is reduced.
(iii) if $I$ is reduced, then $I^{\prime}$ is reduced, and moreover if $\{a, b\}$ is the representation of $I$ such that $a>0$ and $\phi=\frac{b+\sqrt{D}}{2 a}$ is reduced, then the Lagrange neighbour and the Gauss neighbour are the same.

Proof. For any representation $\{a, b\}$ of any primitive ideal, we have

$$
\begin{equation*}
\left|\frac{\sqrt{D}-b}{2 c}\right|\left|\frac{\sqrt{D}+b}{2 a}\right|=1 . \tag{8.3}
\end{equation*}
$$

Now take $\{a, b\}$ to be a half-reduced representation of the half-reduced ideal $I$ so that $0<\frac{-b+\sqrt{D}}{2|c|}<1$, where $c=\left(D-b^{2}\right) / 4 a$.
(i) Suppose that $b<-\sqrt{D}$. Then we have $b^{2}-D=4|a||c|$ so that (8.3) becomes $\left(\frac{\sqrt{D}-b}{2|c|}\right)\left(\frac{-b-\sqrt{D}}{2|a|}\right)=1$. As $0<\frac{-b+\sqrt{D}}{2|c|}<1$, we see that $\frac{-b-\sqrt{D}}{2|a|}>1$. But, as $\left\{a^{\prime}, b^{\prime}\right\}$ is also half-reduced, we have $\frac{-b^{\prime}+\sqrt{D}}{2|a|}<1$, so that $-b^{\prime}+\sqrt{D}<2|a|<-b-\sqrt{D}$, proving that $b^{\prime}>b+2 \sqrt{D}$.
(ii) Suppose that $b>-\sqrt{D}$. Then, we have $|b|<\sqrt{D}$, and (8.3) can be written

$$
\left(\frac{\sqrt{D}-b}{2|c|}\right)\left(\frac{\sqrt{D}+b}{2|a|}\right)=1
$$

showing that $\frac{\sqrt{D}+b}{2|a|}>1$. Or the other hand, as $\left\{a^{\prime}, b^{\prime}\right\}$ is half-reduced, we have $0<\frac{\sqrt{D}-b^{\prime}}{2|a|}<1$, that is $0<\frac{\sqrt{D}+b}{2|a|}-\frac{a}{|a|} q<1$, so that

$$
\frac{a}{|a|} q=\left[\frac{\sqrt{D}+b}{2|a|}\right] \geqslant 1 .
$$

Hence we obtain

$$
\sqrt{D}+b^{\prime}=\sqrt{D}-b+2 a q=(\sqrt{D}-b)+2|a|\left(\frac{a q}{|a|}\right)>2|a|
$$

which, together with the inequalities $0<\frac{\sqrt{D}-b^{\prime}}{2|a|}<1$, shows that $\phi^{\prime}$ is reduced if $a>0$ and $-\phi^{\prime}$ is reduced if $a<0$, proving that $I^{\prime}$ is reduced. (iii) We suppose that $I$ is reduced and choose the representation $\{a, b\}$ of $I$ with $a>0$ and $\phi=\frac{b+\sqrt{D}}{2 a}$ reduced. As $\phi$ is half-reduced and $b>-\sqrt{D}$ from (ii) we see that $I^{\prime}$ is reduced. Moreover, the integer $q$ used to obtain both the Lagrange neighbour and the Gauss neighbour of $\{a, b\}$ is [ $\phi]$. This shows that the two neighbours of $\{a, b\}$ are the same and concludes the proof of Proposition 11.

Definition 15. (Gauss's reduction process ([1]: §§ 183-185)) We start with
a primitive ideal $I_{0}$ of $O_{D}$ and a representation $\{a, b\}$ of $I_{0}$, and define the sequence of representations $\left\{a_{n}, b_{n}\right\}$ of the primitive ideals $I_{n}$ by

$$
\left\{a_{n}, b_{n}\right\} \xrightarrow{G}\left\{a_{n+1}, b_{n+1}\right\} \quad(n=0,1,2, \ldots) .
$$

We now show that Gauss's reduction process leads to a reduced ideal equivalent to $I_{0}$. In addition we give an upper bound for the number of steps required to obtain a reduced ideal $I_{n}$ as well as bounds for a quantity $\rho$ in the relation $I_{n}=\rho I_{0}$.

Proposition 12. (i) The ideal $I_{n}$ is reduced for

$$
n>\max \left(\frac{\left|a_{0}\right|}{\sqrt{D}}+1,2\right)
$$

(ii) Let $I^{\prime}$ be the first reduced ideal obtained by applying Gauss's reduction to $I_{0}$. Then $I=\rho I_{0}$ with $\frac{1}{\left|a_{0}\right|} \leqslant \rho<\sqrt{D}$.

Proof. We suppose that $n>\max \left(\frac{\left|a_{0}\right|}{\sqrt{D}}+1,2\right)$ so that $n \geqslant 3$.
If $b_{1}>-\sqrt{D}$, by Proposition 11 (ii), $I_{2}$ is reduced and so, by Proposition 11 (iii), $I_{n}$ is reduced.

Suppose on the other hand that $b_{1}<-\sqrt{D}$ and that $I_{n}$ is not reduced. Then, by Proposition 11 (ii), we see that $b_{i}<-\sqrt{D}$ for $i=1,2, \ldots, n-1$. Then, by Proposition 11 (i), we have

$$
b_{n-1}>b_{1}+2(n-2) \sqrt{D} .
$$

Hence we obtain

$$
\begin{aligned}
b_{n-1} & >-b_{0}+2 a_{0}\left(\frac{a_{0}}{\left|a_{0}\right|}\left[\frac{a_{0}}{\left|a_{0}\right|} \frac{\left(b_{0}+\sqrt{D}\right.}{2 a_{0}}\right]\right)+2\left(\frac{\left|a_{0}\right|}{\sqrt{D}}-1\right) \sqrt{D} \\
& >-b_{0}+2\left|a_{0}\right|\left(\frac{b_{0}+\sqrt{D}}{2\left|a_{0}\right|}-1\right)+2\left(\frac{\left|a_{0}\right|}{\sqrt{D}}-1\right) \sqrt{D} \\
& =-\sqrt{D},
\end{aligned}
$$

which is a contradiction. This completes the proof that $I_{n}$ is reduced for $n>\max \left(\frac{\left|a_{0}\right|}{\sqrt{D}}+1,2\right)$.
(ii) Let $I_{n}$ be the first reduced ideal obtained from $I_{0}$ by Gauss's reduction
process. If $n=0$ then $\rho=1$, so that $\frac{1}{\left|a_{0}\right|} \leqslant \rho<\sqrt{D}$. If $n \geqslant 1$ we have $I_{n}=\rho I_{0}$ with (by (2.12))

$$
\rho=\left|\frac{a_{1}}{a_{0}} \phi_{1} \ldots \frac{a_{n}}{a_{n-1}} \phi_{n}\right|=\left|\frac{a_{n}}{a_{0}}\right|\left|\frac{b_{1}+\sqrt{D}}{2 a_{1}}\right| \ldots\left|\frac{b_{n}+\sqrt{D}}{2 a_{n}}\right| .
$$

As the representations $\left\{a_{k}, b_{k}\right\}$ are half-reduced for $k \geqslant 1$, we see, by (8.3), that $\left|\frac{b_{k}+\sqrt{D}}{2 a_{k}}\right|>1(k \geqslant 1)$ so that $\rho>\left|\frac{a_{n}}{a_{0}}\right| \geqslant \frac{1}{\left|a_{0}\right|}$. On the other hand we have

$$
\rho=\left|\frac{b_{1}+\sqrt{D}}{2 a_{0}}\right| \ldots\left|\frac{b_{n}+\sqrt{D}}{2 a_{n-1}}\right| .
$$

As $\left\{a_{k}, b_{k}\right\}$ is a half-reduced representation for $k=1,2, \ldots, n$, we have $0<\sqrt{D}-b_{k}<2\left|a_{k-1}\right|$. Furthermore, for $k=1,2, \ldots, n-1$, we have $\sqrt{D}+b_{k}<2\left|a_{k-1}\right|, \quad$ as otherwise $0<\sqrt{D}-b_{k}<2\left|a_{k-1}\right|<\sqrt{D}+b_{k}$, which is equivalent to $0<\sqrt{D}-b_{k}<2\left|a_{k}\right|<\sqrt{D}+b_{k}$ so that by (4.2) the primitive ideal $I_{k}$ would be reduced. Therefore, for $k=1,2, \ldots, n-1$, we have

$$
\left|\sqrt{D}+b_{k}\right| \leqslant \sqrt{D}+\left|b_{k}\right|=\left\{\begin{array}{lll}
\sqrt{D}+b_{k}<2\left|a_{k-1}\right|, & \text { if } & b_{k} \geqslant 0, \\
\sqrt{D}-b_{k}<2\left|a_{k-1}\right|, & \text { if } & b_{k}<0,
\end{array}\right.
$$

so that, as $\left\{a_{n}, b_{n}\right\}$ is reduced,

$$
\rho<\frac{b_{n}+\sqrt{D}}{2\left|a_{n-1}\right|}<\sqrt{D}
$$

which completes the proof of Proposition 12.
We remark that Proposition 7 and 12 suggest that Lagrange's reduction process may lead to a reduced ideal much faster than Gauss's reduction process, as the number $M_{0}$ of Lemma 6 is much smaller than $\max \left(\frac{\left|a_{0}\right|}{\sqrt{D}}+1,2\right)$.

Example 5. We apply both Lagrange reduction and Gauss reduction to the representation $\{3655,7068\}$ of the primitive ideal $[3655,3534+\sqrt{21}]$ of $O_{84}$. We obtain

$$
\{3655,7068\} \xrightarrow{L}\{-3417,-7068\} \xrightarrow{L}\{4,234\} \xrightarrow{L}\{3,6\} \quad \text { (3 steps) }
$$

and

$$
\begin{aligned}
&\{3655,7068\} \xrightarrow{G}\{-3417,-7068\} \\
& \xrightarrow{G}\{3187,-6600\} \xrightarrow{G}\{-2965,-6148\} \xrightarrow{G} \ldots \\
& \xrightarrow{G}\{-1,-12\} \\
& \xrightarrow{G}\{-5,8\} \quad(30 \text { steps }) .
\end{aligned}
$$

We remark that $M_{0}$ is approximately 8.72 and $\frac{\left|a_{0}\right|}{\sqrt{D}}+1$ is approximately 399.8 .

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