# 2. Conjugate points

A *geodesic* will always be a locally distance-realizing curve parametrized proportionally to arclength by [0, 1]. A geodesic space is *locally convex* if every point has a neighborhood such that the distance $d(\alpha(t), \beta(t))$ is convex for any two minimizing geodesics $\alpha$ and $\beta$ in the neighborhood. (Of course, for Riemannian manifolds without boundary this is equivalent to nonpositive sectional curvature; see [BGS].) If $m_{pq}$ denotes the midpoint of a geodesic from $p$ to $q$, then it is equivalent to say that $M$ is covered by neighborhoods $U$ such that the relation

$$2d(m_{pq}, m_{pr}) \leqslant d(q, r)$$

holds for any three points $p$, $q$ and $r$ in $U$ and any geodesics in $U$ joining them (such geodesics are unique).

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We also thank the referee for informing us of the chapter [Ba] by W. Ballmann that is to appear in *Sur les Groupes Hyperboliques d'après Gromov* (Ghys, de la Harpe, eds.), and its author for promptly sending us a preprint. In [Ba], the Hadamard-Cartan theorem is proved using the Birkhoff curve-shortening technique; this depends on local compactness, which we avoid by exploiting local convexity. Another distinction is that the notions of exponential map and conjugate point are not introduced in [Ba]. The Hadamard-Cartan theorem is applied in [Ba] to obtain a criterion for the hyperbolicity of certain simply connected polyhedra.

## 2. Conjugate points

In a given geodesic space, let $\mathbf{G}_m$ be the space of geodesics starting at $m$, carrying the uniform metric $\mathbf{d}$. Say the point $m$ has *no conjugate points* if the endpoint map on $\mathbf{G}_m$ maps some neighborhood of every $\gamma$ homeomorphically onto a neighborhood of the endpoint of $\gamma$. (In Riemannian manifolds without boundary, this definition is equivalent to the usual one.)

THEOREM 2. *A locally convex, complete geodesic space has no conjugate points.*

Here it is straightforward that the endpoint map is a homeomorphism, and in fact an isometry, from some open neighborhood of every $\gamma$ onto its image. The question is whether it is surjective; that is, whether locally there always exist geodesics from $m$ that vary continuously with their righthand endpoints.

To show this, especially in the absence of local compactness, seems to require a little care.

*Proof.* Fix a geodesic $\gamma$. The maximum radius of an open ball, such that the distance function is convex between minimizing geodesics with endpoints in the ball, is positive and varies continuously with the center of the ball. Thus the infimum, $r$, of these radii over all the points of $\gamma$ is positive. Suppose $\alpha_1$ and $\alpha_2$ are geodesics whose distances from $\gamma$, namely

$$\mathbf{d}(\alpha_i, \gamma) = \max d(\alpha_i(t), \gamma(t)) \ ,$$

are less than $r$. Since convexity is a local property, the distance function $d(\alpha_1(t), \alpha_2(t))$ is convex, and hence the larger of its two endpoint values is $\mathbf{d}(\alpha_1, \alpha_2)$.

Let $P(L)$ be the statement:

> For every subsegment $\bar{\gamma}$ of $\gamma$ of length at most $L$, any two points $p$ and $q$ whose respective distances from the endpoints of $\bar{\gamma}$ are less than $r/2$ are joined by a geodesic $\alpha = \alpha(p,q)$ whose distance from $\bar{\gamma}$ is less than $r/2$.

Note that $\alpha(p,q)$ is necessarily unique, and the distance function between any two such geodesics is convex.

Clearly $P(r)$ holds. We claim that $P(3L/2)$ holds if $P(L)$ does. Indeed, suppose $\bar{p}$ and $\bar{q}$ are the left and right endpoints of a subsegment of $\gamma$ of length at most $3L/2$, and let $p_0$ and $q_0$ trisect its length, moving from left to right. Suppose $p$ and $q$ lie within distance $R < r/2$ of the endpoints $\bar{p}$ and $\bar{q}$, respectively. Applying $P(L)$ to $\alpha(\bar{p}, q_0)$ and $\alpha(p_0, \bar{q})$ repeatedly, we define $p_i$ and $q_i$ inductively for $i \geq 1$ by letting $p_i$ be the midpoint of $\alpha(p, q_{i-1})$ and $q_i$ be the midpoint of $\alpha(p_{i-1}, q)$. Convexity ensures that $d(p_{i-1}, p_i)$ and $d(q_{i-1}, q_i)$ do not exceed $R/2^i$ and hence do not exceed $r/2^{i+1}$. Therefore the sequences $\{p_i\}$ and $\{q_i\}$ are Cauchy, converging respectively to points $p_\infty$ and $q_\infty$ within distance $R$ of $p_0$ and $q_0$. Since the distance function between $\alpha(p, q_i)$ and $\alpha(p, q_\infty)$ is convex, $\{\alpha(p, q_i)\}$ converges uniformly to $\alpha(p, q_\infty)$. Similarly, $\{\alpha(p_i, q)\}$ converges to $\alpha(p_\infty, q)$. Each of these limit geodesics contains a reparametrization of $\alpha(p_\infty, q_\infty)$, so they combine to give the desired geodesic joining $p$ and $q$.

We conclude, in particular, that the endpoint map sends the ball of radius $r/2$ about a geodesic $\gamma$ in $\mathbf{G}_m$ isometrically onto the ball of the same radius about the endpoint of $\gamma$. □

Now we indicate how the above argument on conjugate points fits into the Alexandrov theory of spaces of curvature bounded above. Following [ABN], we shall say that a geodesic space has *curvature bounded above by $K$* if every point has a "model neighborhood" in which any two points are joined by a minimizing geodesic in the neighborhood, and any minimizing geodesic triangle in the neighborhood has perimeter less than $2\pi/\sqrt{K}$ (if $K > 0$), and angle sum at most equal to the sum for a triangle having the same sidelengths in the standard surface $S_K$ of constant curvature $K$. Alexandrov proved that then each angle individually is at most equal to its comparison angle in $S_K$ [A1]. Here the *angle* at a vertex of a given minimizing geodesic triangle is defined to be the lim sup of the corresponding comparison angles in $S_K$ over all triangles obtained by approaching the vertex along its adjacent sides. Curvature bounded above by 0 is a stronger condition in general than local convexity [A1]; for instance, most Minkowski spaces satisfy the latter and not the former.

The Alexandrov development method then shows that minimizing geodesics in a model neighborhood are unique and vary continuously with their endpoints ([A2], p. 51-56). The main step in this method is the proof that if one forms a triangle in a model neighborhood by moving distances $x$ and $y$ along two minimizing geodesics from $m$, then the angle in the model triangle in $S_K$ at the point corresponding to $m$ is nondecreasing in $x$ and $y$. It follows from this by a hinge argument that the distance between any two points of a triangle in a model neighborhood is no greater than the distance in $S_K$ between the two corresponding points of the model triangle. Alexandrov further proves that in any region in which minimizing geodesics are unique and vary continuously with their endpoints, the angle comparison property for minimizing geodesic triangles holds globally as well as locally ([A2], p. 56-58). Alexandrov's development method may also be applied to an arbitrary, not necessarily minimizing, geodesic $\gamma$ in $\mathbf{G}_m$ (of length less than $\pi/\sqrt{K}$ if $K > 0$), and any two geodesics $\sigma_1$ and $\sigma_2$ sufficiently close in $\mathbf{G}_m$ to $\gamma$. We outline the argument.

We may assume that $\sigma_1$ and $\sigma_2$ lie within distance $r/2$ of $\gamma$, where $r$ is a uniform model radius for $\gamma$, and that all geodesic triangles $\Delta(t) = m\sigma_1(t)\sigma_2(t)$ consisting of subsegments of $\sigma_1$ and $\sigma_2$ and the minimizing geodesic between their righthand endpoints have perimeters less than $2\pi/\sqrt{K}$. For all $t$ in some interval $[0, t_0]$, the sidelengths of $\Delta(t)$ satisfy the triangle inequalities, $\Delta(t)$ and its model triangle in $S_K$ satisfy the angle comparison property, and the angle $\theta(t)$ at the point corresponding to $m$ in the model triangle is nondecreasing in $t$. We claim that these properties extend to $\min\{t_0 + \varepsilon, 1\}$, for uniform $\varepsilon$, whenever they extend to $t_0 < 1$; and hence they extend to $t_0 = 1$. To see this, choose $\varepsilon$ so that the restrictions of $\sigma_1$ and $\sigma_2$ to $[t_0, t_0 + \varepsilon]$ lie in a model neighborhood of $\gamma(t_0)$ and are minimizing. Construct a pentagon in $S_K$ in the obvious way out of three model triangles corresponding to $\Delta(t_0)$, $\Delta\sigma_1(t_0)\sigma_1(u)\sigma_2(t_0)$ and $\Delta\sigma_2(t_0)\sigma_2(u)\sigma_1(u)$ respectively. By the angle comparison property, the interior angles of this pentagon at the points corresponding to $\sigma_1(t_0)$ and $\sigma_2(t_0)$ are at least $\pi$. Thus this pentagon determines a surface with boundary in $S_K$ whose boundary is itself a minimizing geodesic triangle in the interior metric. Therefore the triangle inequalities hold for $\Delta(u)$. By straightening the two concave sides of the pentagon one increases the three convex angles and hence obtains a model triangle for $\Delta(u)$ that satisfies the angle comparison property. The same argument applied to $\Delta(u)$ and $\Delta(v)$ for $t_0 \leqslant u < v \leqslant t_0 + \varepsilon$ shows that $\theta(u) \leqslant \theta(v)$, and hence verifies the above claim. It follows by a hinge argument that $\Delta(1)$ satisfies the following *uniform distance comparison property:* for $0 \leqslant t \leqslant 1$, the distance between $\sigma_1(t)$ and $\sigma_2(t)$ is no greater than the distance in $S_K$ between the corresponding points of the model triangle for $\Delta(1)$. In particular, the endpoint map on $\mathbf{G}_m$ is injective on a neighborhood of $\gamma$.

One may then ask whether the endpoint map is surjective, sending a neighborhood of $\gamma$ onto a neighborhood of its endpoint. To answer this question fully, we indicate how to extend the proof of Theorem 2 to the case $K > 0$:

THEOREM 3. *A complete geodesic space of curvature bounded above by $K > 0$ has no conjugate points along geodesics of length less than* $\pi/\sqrt{K}$.

It is easy to give examples showing that local injectivity of the endpoint map may not imply local surjectivity beyond length $\pi/\sqrt{K}$. (However, in Riemannian manifolds without boundary, local injectivity of the exponential map implies regularity and hence local surjectivity [W].) For instance, a closed unit hemisphere in its interior metric has curvature bounded above by 1; here the nature of $\mathbf{G}_m$ changes abruptly at length $\pi$. If $\gamma$ lies on the boundary circle and has length $\pi + \varepsilon$, then a small neighborhood of $\gamma$ is mapped homeomorphically onto a circular segment, not onto a neighborhood of the endpoint.

*Proof.* Fix a geodesic $\gamma$ of length less than $\pi/\sqrt{K}$. Let $r > 0$ be a uniform radius for model balls around points of $\gamma$. Let $P(L)$ be the statement:

Given $\varepsilon$ in $(0, r)$, there is $\delta > 0$ such that for every subsegment $\bar{\gamma}$ of $\gamma$ of length at most $L$, any two points $p$ and $q$ whose respective distances from the endpoints of $\bar{\gamma}$ are less than $\delta$ are joined by a geodesic $\alpha = \alpha(p, q)$ whose distance from $\bar{\gamma}$ is less than $\varepsilon$ and whose length is at most $L + \varepsilon$.

If $\rho = \min\{r/2, \pi/4\sqrt{K}\}$, then $P(\rho)$ holds, by taking $\delta = \min\{\varepsilon/2, \rho/2\}$. This estimate uses the the fact that the distances between corresponding points of two sides of a triangle in $S_K$ never exceed the endpoint value if both sides have length less than $\pi/2\sqrt{K}$. It remains to be shown that if $P(L)$ holds and $L < 2\pi/3\sqrt{K}$, then $P(3L/2)$ holds.

Choose $\varepsilon < \min\{r/2, \pi/6\sqrt{K}\}$. This choice of $\varepsilon$ ensures that any two geodesics issuing from the same point and having length at most $L + \varepsilon$ and distance at most $\varepsilon$ from a subsegment of $\gamma$ will satisfy the uniform distance comparison property. In particular, the geodesics $\alpha(p, q)$ in $P(L)$ are unique. Now choose $\varepsilon' < \min\{2\pi/3\sqrt{K} - L, 2\varepsilon/3\}$. Denote by $\delta'$ the value given by applying $P(L)$ to $\gamma$, with $\varepsilon'$ as the desired distance from subsegments of $\gamma$. Set $L' = L + \varepsilon'$ and $\lambda = \sin(\sqrt{K}L'/2)/\sin(\sqrt{K}L')$ (then $1/2 < \lambda < 1$). It is an exercise in spherical trigonometry to show that if $\beta_1$ and $\beta_2$ are two sides of a minimizing triangle in $S_K$ and both have length less than $L'$, then

$$d(\beta_1(1/2), \beta_2(1/2)) < \lambda d(\beta_1(1), \beta_2(1)) \,. \tag{1}$$

Let $\delta = (1 - \lambda)\delta'/\lambda$ (then also $\delta < \delta'$).

Suppose that $\bar{\gamma}$ is a subsegment of $\gamma$ of length $\bar{L} \leqslant 3L/2$, with endpoints $\bar{p}$ and $\bar{q}$, and let $p$ and $q$ be points within distance $\delta$ of these endpoints. We now follow the construction of Theorem 2. Subdivide $\bar{\gamma}$ into thirds by points $p_0$, $q_0$ and take, recursively, $p_i$ as the midpoint of $\alpha(p, q_{i-1})$ and $q_i$ as the midpoint of $\alpha(p_{i-1}, q)$. To verify that this recursive definition is possible, apply $P(L)$ repeatedly to the subsegments $\alpha(\bar{p}, q_0)$ and $\alpha(p_0, \bar{q})$ of $\gamma$, and note that inductively $d(p_{i-1}, p_i)$ and $d(q_{i-1}, q_i)$ are less than $\lambda^i\delta$ by the uniform distance comparison property and (1), and hence $d(p_0, p_i)$ and $d(q_0, q_i)$ are less than $\lambda\delta/(1 - \lambda) = \delta'$. In particular, $\{p_i\}$ and $\{q_i\}$ are Cauchy, and converge to $p_\infty$ and $q_\infty$ respectively. By the uniform distance comparison property, $\{\alpha(p, q_i)\}$ converges uniformly to $\alpha(p, q_\infty)$ and $\{\alpha(p_i, q)\}$ to $\alpha(p_\infty, q)$. These two limit geodesics overlap since $\alpha(p_\infty, q_\infty)$ is unique, hence combine to give a geodesic from $p$ to $q$ that has distance at most $\varepsilon' < \varepsilon$ from $\bar{\gamma}$ and length at most $\bar{L} + 3\varepsilon'/2 < \bar{L} + \varepsilon$. □

## 3. Proof of Theorem 1

Again consider a locally convex, complete geodesic space $M$, and let $\mathbf{G}_m$ be the space of geodesics starting at $m$ carrying the uniform metric $\mathbf{d}$. It follows from local convexity that a Cauchy sequence in $\mathbf{G}_m$ converges to a geodesic and hence $\mathbf{d}$ is complete. Furthermore, $M$ has *neighborhoods of bipoint uni-*