

# THE GEOMETRIC CONSTRUCTION

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## THE CANTOR SET AND A GEOMETRIC CONSTRUCTION

by Marco PAVONE

### INTRODUCTION

The Cantor ternary set consists of all those real numbers  $x$  in  $[0, 1]$  which have a ternary expansion  $x = \sum_{n=1}^{\infty} a_n/3^n$  for which  $a_n$  is never 1. Equivalently,  $C$  can be obtained in a purely geometrical fashion by first removing from  $[0, 1]$  the middle third  $(1/3, 2/3)$ , then removing the middle thirds  $(1/9, 2/9)$  and  $(7/9, 8/9)$  of the remaining intervals, and so on ( $C$  will be exactly the complement of the countable union of the removed intervals). If  $x = \sum_{n=1}^{\infty} a_n/3^n$  is in  $C$ , the geometric interpretation of its ternary expansion is that  $x$  is the unique point in  $[0, 1]$  which is reached by first staying to the left or to the right of  $(1/3, 2/3)$  if  $a_1 = 0$  or  $a_1 = 2$  respectively, then staying to the left or to the right of the next removed interval if  $a_2 = 0$  or  $a_2 = 2$  respectively, and so on. It follows from the construction that  $C$  is a nowhere dense closed subset of  $[0, 1]$ .

A well known property of  $C$  is that any real number in  $[0, 2]$  can be written as the sum of two numbers in  $C$ . The purpose of this note is to give an elementary proof of  $C + C = [0, 2]$  which only uses the geometric definition of  $C$ . A refinement of the proof shows in fact that for any  $k$  in  $[0, 2]$  there exists either a finite or an uncountable number of pairs  $x, y$  from  $C$  such that  $x + y = k$ . We also discuss the analogy between this decomposition result and certain properties of continued fractions.

### THE GEOMETRIC CONSTRUCTION

We set, as usual,  $C \times C = \{(x, y) \in \mathbf{R}^2 : x, y \in C\}$ . Then  $C + C = [0, 2]$  can be geometrically restated as

(\*) for any  $k$  in  $[0, 2]$  the line  $x + y = k$  intersects  $C \times C$  in at least one point.

Let's agree to call a line segment in  $\mathbf{R}^2$  "horizontal" or "vertical" if it is parallel or perpendicular to the line  $y = x$  respectively. Consider a sequence  $L_0, L_1, L_2, \dots$  of continuous polygonal curves in  $\mathbf{R}^2$  with the following properties (see fig. 1-3):

- (a)  $L_n$  is contained in  $[0, 1] \times [0, 1]$  for all  $n$ , and is composed by horizontal and vertical segments only.
- (b) The vertices of  $L_n$  belong to  $C \times C$  for all  $n$ .
- (c) The endpoints of  $L_n$  are  $(0, 0)$  and  $(1, 1)$  for all  $n$ .
- (d) Each  $L_n$  contains  $3^n$  horizontal segments, each of which has length  $2^{1-2} 3^n$ .
- (e) For all  $n$ , and for any  $k$  in  $\{0, 2 \cdot 3^n, 4 \cdot 3^n, \dots, 2\}$  the line  $x + y = k$  contains a vertical segment of  $L_n$ .
- (f) For all  $n$ , and for any  $k$  not in  $\{0, 2 \cdot 3^n, 4 \cdot 3^n, \dots, 2\}$  the line  $x + y = k$  meets at most one horizontal segment of  $L_n$ .

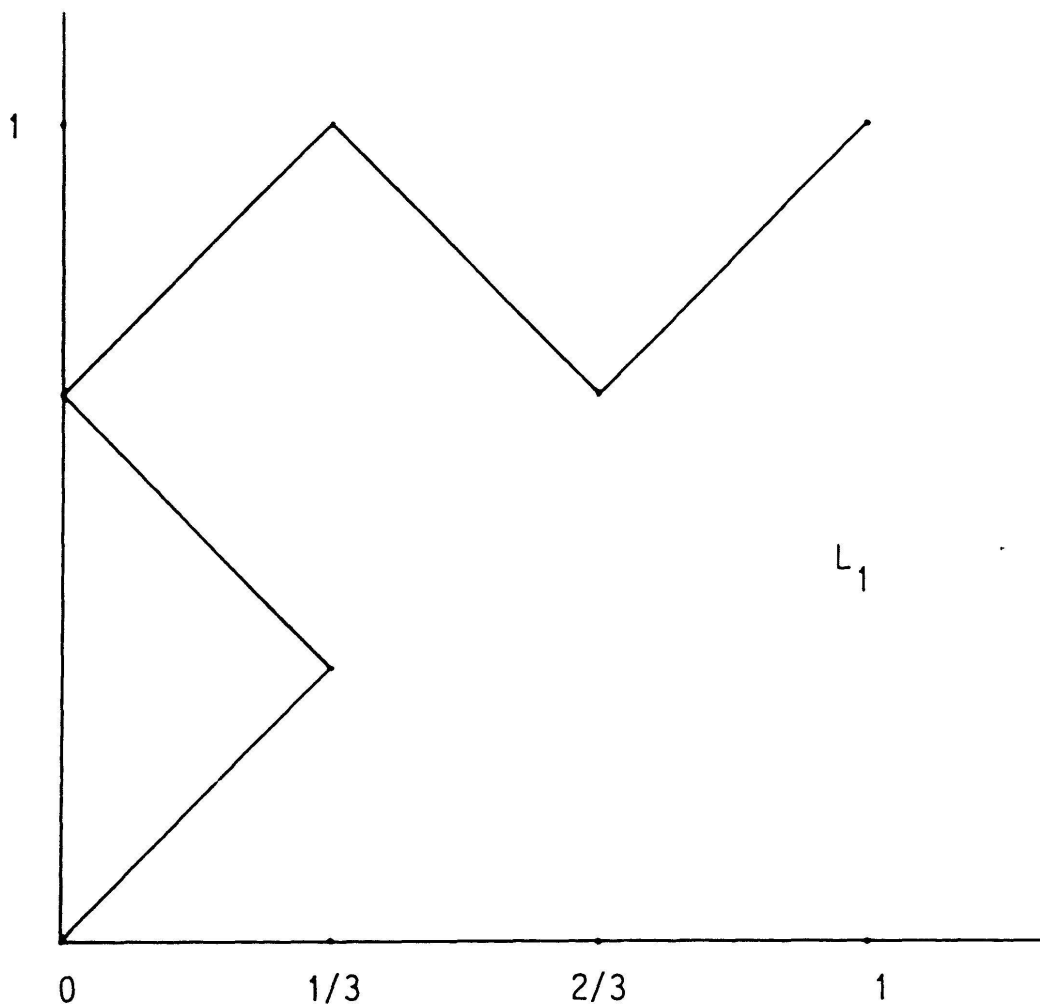


FIGURE 1

Suppose first that such a sequence exists. Then property (\*) is satisfied. Indeed, fix  $k$  in  $[0, 2]$  and let  $r$  denote the line  $x + y = k$ . If  $k$  is in  $\{0, 2/3^n, 4/3^n, \dots, 2\}$  for some  $n$ , then  $r$  meets  $C \times C$  by (e) and (b); otherwise, for any positive integer  $n$  there exists by (f) a unique horizontal segment of  $L_n$  that meets  $r$ . This implies, by (d) and (b), that  $\text{dist}(r, C \times C) < 2^{1/2}/3^n$  for all positive integers  $n$ , that is,  $\text{dist}(r, C \times C) = 0$ . Then  $r$  meets  $C \times C$  by a standard compactness argument (I recall that  $C$  is a closed subset of  $[0, 1]$ ).

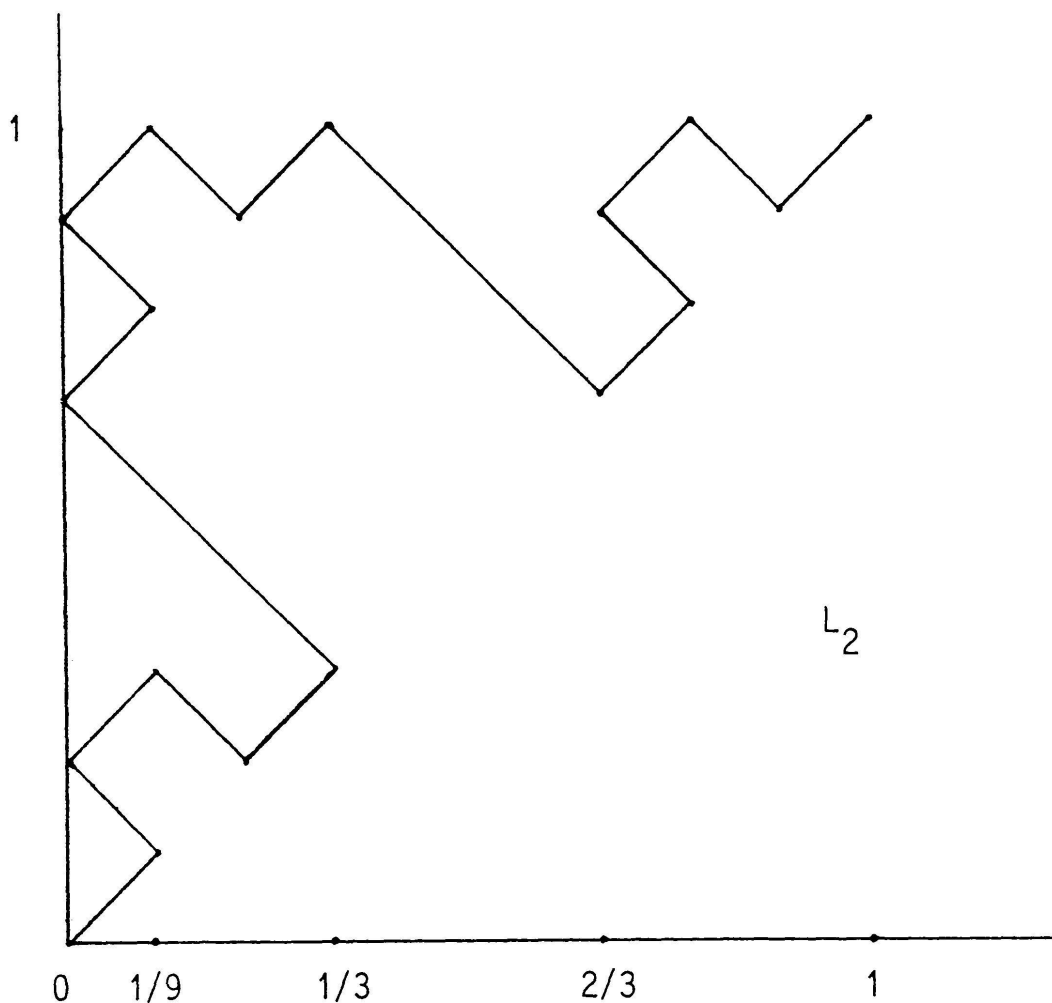


FIGURE 2

We now proceed to the heart of the argument, that is the construction of the sequence  $\{L_n\}_n$ . All we need is in fact the first step of an induction process. Let  $L_0$  be the line segment with endpoints  $(0, 0)$  and  $(1, 1)$ , and let  $L_1$  be the polygonal with vertices  $(0, 0)$ ,  $(1/3, 1/3)$ ,  $(0, 2/3)$ ,  $(1/3, 1)$ ,  $(2/3, 2/3)$  and  $(1, 1)$  (see fig. 1). In general, let  $L_{n+1}$  be the curve obtained from  $L_n$  by performing on each horizontal segment of  $L_n$  the same modification that was performed on  $L_0$  to get  $L_1$ . In other words, we replace the generic

horizontal segment of  $L_n$  with endpoints  $(x, y)$  and  $(x + 1/3^n, y + 1/3^n)$  by the polygonal passing through the points

$$(x, y), \quad (x + 1/3^{n+1}, y + 1/3^{n+1}), \quad (x, y + 2/3^{n+1}), \quad (x + 1/3^{n+1}, y + 1/3^n), \\ (x + 2/3^{n+1}, y + 2/3^{n+1}) \quad \text{and} \quad (x + 1/3^n, y + 1/3^n)$$

(see fig. 2 and 3). It is then apparent that  $\{L_n\}_n$  satisfies the hypotheses (a), ..., (f) stated above.

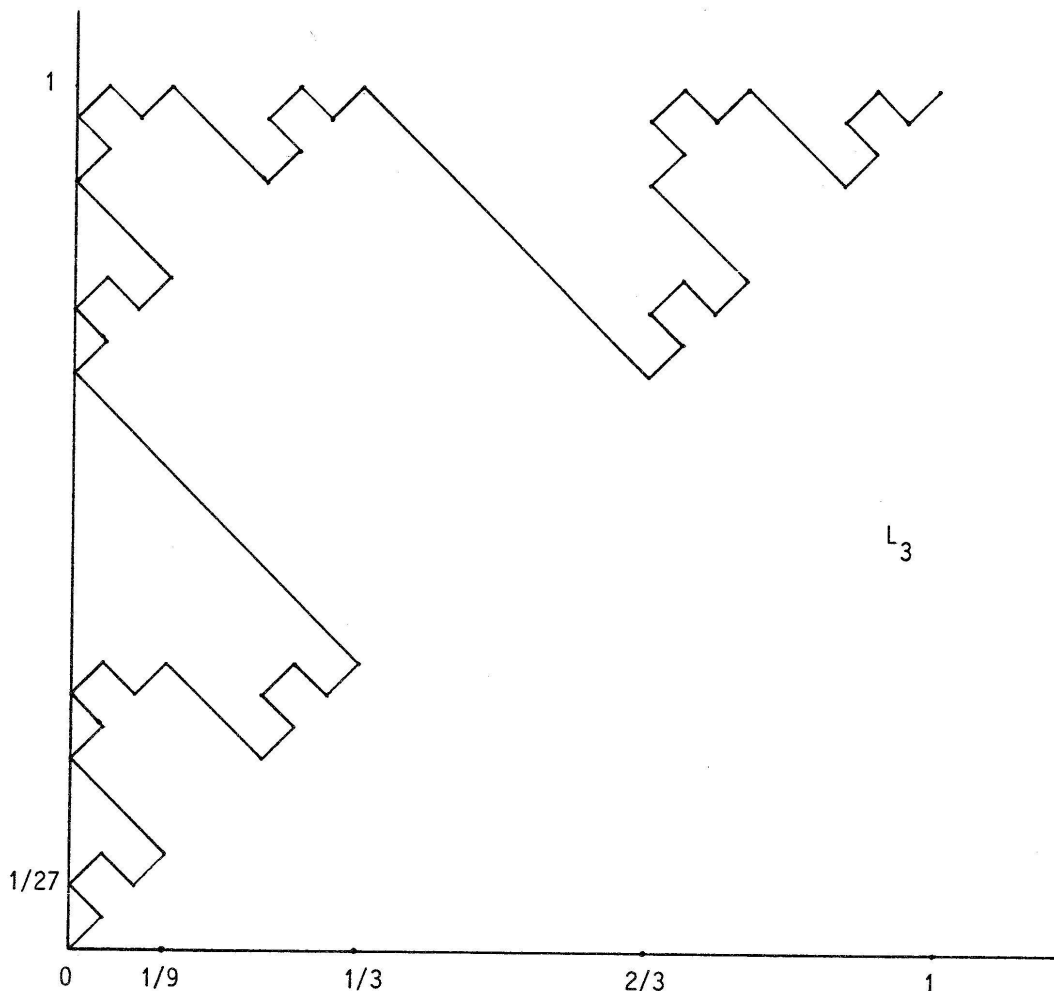


FIGURE 3

An easy modification of the previous construction gives us more information on the way a number in  $[0, 2]$  can be written as the sum of two numbers in  $C$ . For every map  $\mu$  from  $\mathbb{N} \setminus \{0\}$  into  $\{0, 2\}$  we construct a sequence  $\{L_n^{(\mu)}\}_n$  of polygonal curves with properties (a), ..., (f). The idea is simply to add to the previous construction a choice between “left” and “right” at every step of the induction. What one ends up with is exactly a two-dimensional version of the geometric construction of the Cantor ternary set. We proceed as follows.

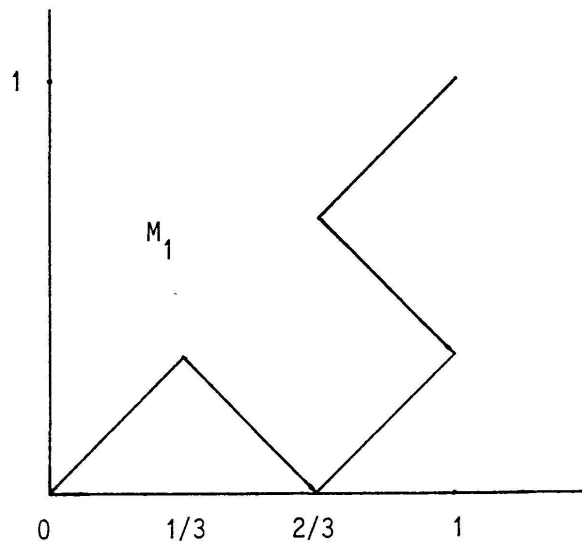


FIGURE 4

Let  $M_1$  be the mirror image of the curve  $L_1$  with respect to the line  $y = x$  (see figure 4). If  $\mu$  is a map from  $\mathbf{N} \setminus \{0\}$  into  $\{0, 2\}$ , we define  $L_0^{(\mu)} = L_0$ , and for any nonnegative integer  $n$  we let  $L_{n+1}^{(\mu)}$  be the polygonal obtained from  $L_n^{(\mu)}$  by replacing each horizontal segment of  $L_n$  by a (normalized) copy of  $L_1$  or  $M_1$ , according to whether  $\mu(n+1) = 0$  or  $\mu(n+1) = 2$  respectively. For example, if  $\mu = \{0, 0, 0, \dots\}$ , we obtain our original sequence  $\{L_n\}_n$  (fig. 1-3), and for  $\mu = \{2, 2, 2, \dots\}$  we get its mirror image with respect to the line  $y = x$ . For  $\mu = \{0, 2, 0, 2, \dots\}$ , we obtain castle-like polygonals as in figure 5.

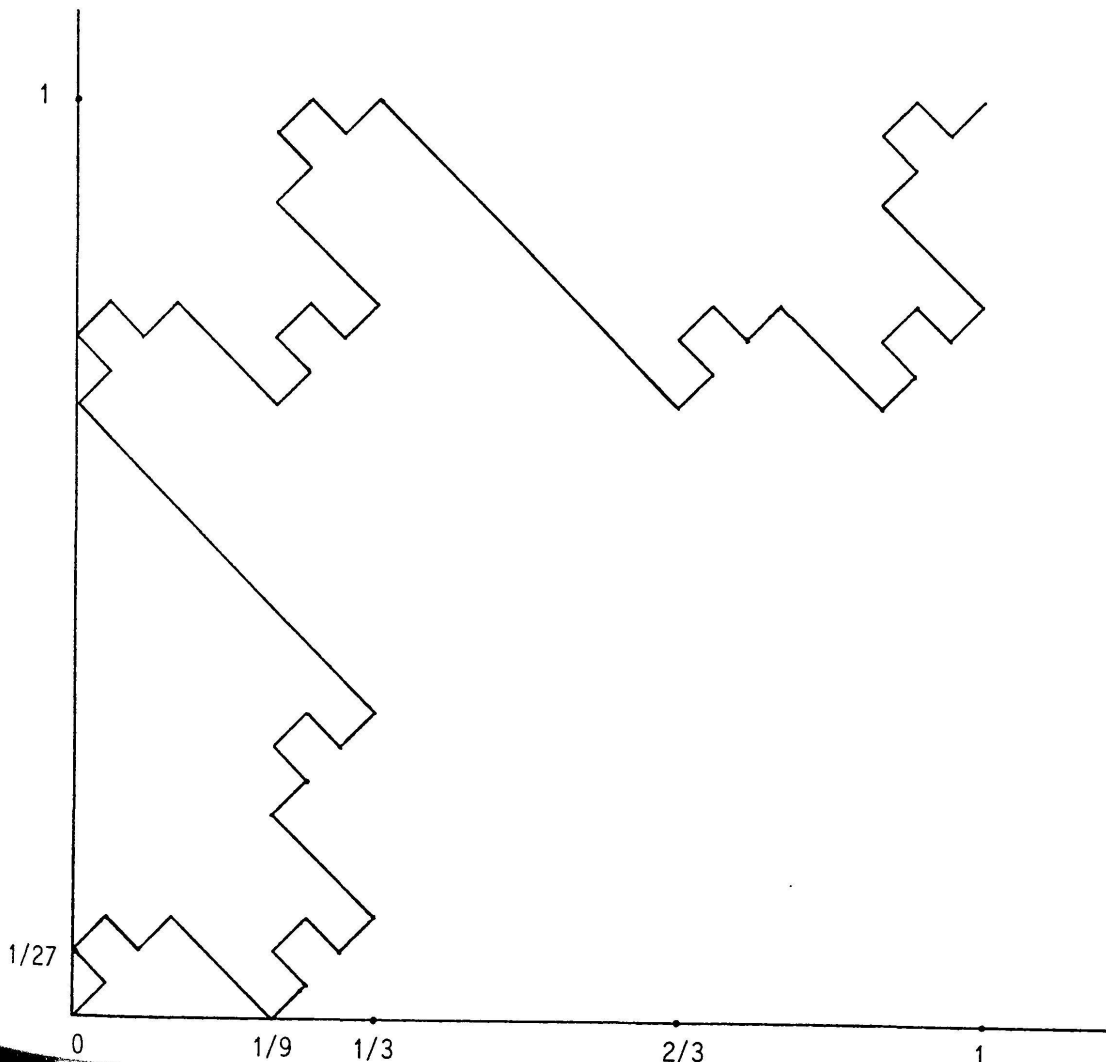


FIGURE 5

For all  $\mu$  let  $L^{(\mu)}$  denote the uniform limit of the curves  $L_n^{(\mu)}$ ,  $n = 0, 1, \dots$ . Then  $L^{(\mu)}$  is a continuous curve in  $[0, 1] \times [0, 1]$  with endpoints  $(0, 0)$  and  $(1, 1)$ , and with the property that, for any  $k$  in  $[0, 2]$ , the line  $x + y = k$  intersects  $L^{(\mu)}$  in some point of  $C \times C$ . Viceversa, given any point  $(x, y)$  in  $C \times C$ , there is some sequence  $\mu$  such that  $(x, y)$  lies on  $L^{(\mu)}$ .

To see this, note that the ternary subdivision of  $[0, 1]$  that generates  $C$  produces a corresponding subdivision of  $[0, 1] \times [0, 1]$  that generates  $C \times C$ . At the  $n$ -th step, the subset  $G_n$  of  $[0, 1] \times [0, 1]$  that contains points of  $C \times C$  is the union of  $4^n$  squares (the black squares in figure 6 for  $n = 3$ ). It is clear that  $G_n$  contains the vertices of the curves  $L_n^{(\mu)}$  for all  $\mu$  (compare figures 3 and 6). The conclusion is now immediate.

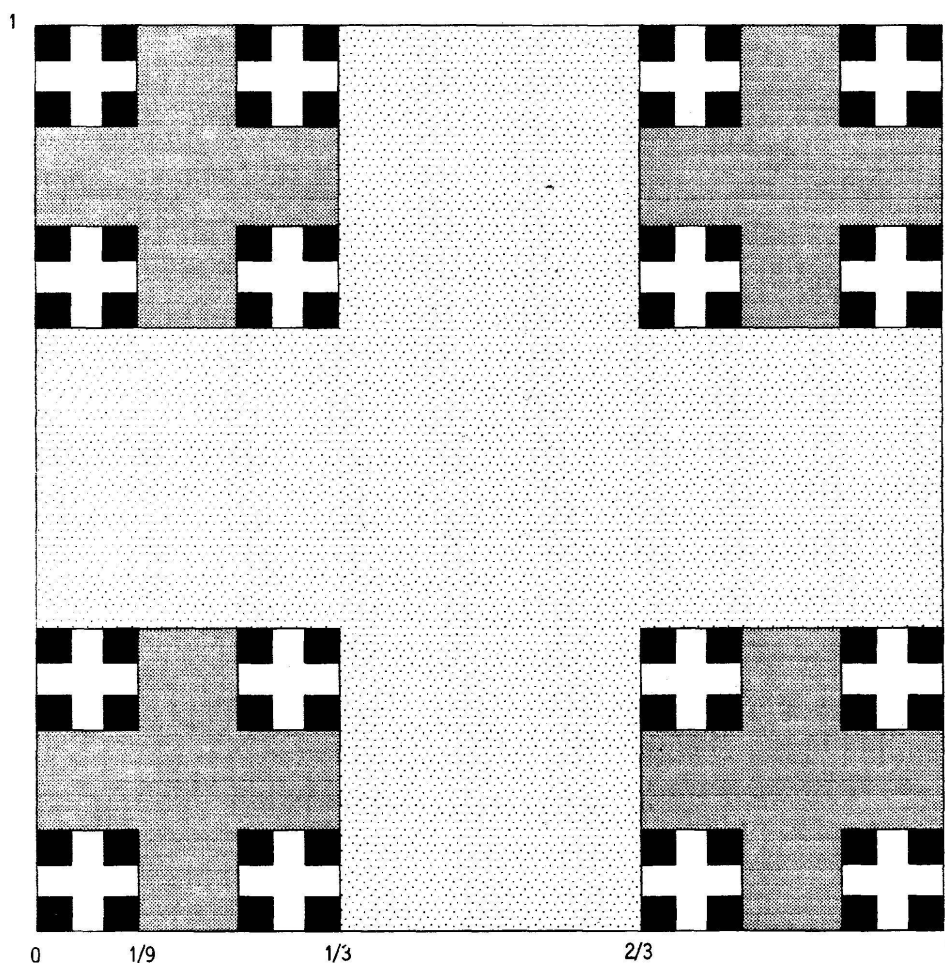


FIGURE 6

Note that if  $\hat{\mu}$  is the sequence obtained from  $\mu$  by turning all the 0's in 2's and viceversa, then the line  $x + y = k$  intersects  $L^{(\mu)}$  in a point  $(x, y)$  if and only if it intersects  $L^{(\hat{\mu})}$  in a point  $(y, x)$ ; in other words,  $\hat{\mu}$  does not give us any new information on the decomposition of  $k$  as a sum of numbers in  $C$ . We shall therefore restrict our attention to sequences  $\mu$  with  $\mu(1) = 0$  (i.e. to curves  $L^{(\mu)}$  above the line  $y = x$ ).

Fix  $k = 2h$  in  $[0, 2]$ ,  $h > 0$ , and let  $h = \sum_{n=1}^{\infty} a_n/3^n$  be the unique infinite ternary expansion of  $h$ . We claim that the equation  $x + y = k$  has a finite or an uncountable number  $S(k)$  of solutions in  $C \times C$  according to whether the cardinality  $c(k)$  of the set  $\{n \in \mathbf{N} \setminus \{0\}; a_n = 1\}$  is finite or infinite respectively. In fact, the exact formula is  $S(k) = 1$  if  $c(k) = 0$  or  $1$ , and  $S(k) = 3(2^{c(k)-2})$  otherwise.

Let  $r$  be the line  $x + y = k$ , and let  $n$  be any positive integer. With the notation set above, and with the help of figure 6, it is easy to see that  $a_n = 1$  if and only if  $G_n$  meets  $r$  in twice as many squares than  $G_{n-1}$ . Equivalently,  $a_n = 1$  if and only if, for all  $\mu$ ,  $r$  meets  $L_{n-1}^{(\mu)}$  in the middle third of one of its horizontal segments; in other words,  $a_n = 1$  if and only if at the  $n$ -th step of the construction the curves  $L_n^{(\mu)}$  meet  $r$  in twice as many points than the curves  $L_{n-1}^{(\mu)}$ . If  $a_n \neq 1$ , the choice between  $\mu(n) = 0$  and  $\mu(n) = 2$  at the  $n$ -th step does not produce any new intersection point. This shows that  $c(k)$  is finite or infinite depending on whether  $r$  meets the curves  $L^{(\mu)}$  in a finite or an uncountable number of points, and our claim is proved.

*Example.* If  $k = 2h = 28/27$  ( $h = 0.11122\dots$  in ternary form, with 2 repeated infinitely often), then  $S(k) = 6$  and the possible decompositions are (in ternary form)  $k = 1 + 0.001$ ,  $k = 0.222 + 0.002$ ,  $k = 0.221 + 0.01$ ,  $k = 0.21 + 0.021$ ,  $k = 0.202 + 0.022$  and  $k = 0.201 + 0.1$ .

In the case where  $c(k)$  is infinite, we saw that each new occurrence of 1 in the sequence  $\{a_n\}_n$  produces a new choice between  $\mu(n) = 0$  and  $\mu(n) = 2$ . In terms of the decomposition  $k = x + y$ , with  $x = \sum_{n=1}^{\infty} b_n/3^n$  and  $y = \sum_{n=1}^{\infty} c_n/3^n$ , this corresponds precisely to choosing  $b_n = c_n = 0$  if  $a_n = 0$ ,  $b_n = c_n = 2$  if  $a_n = 2$ , and finally  $b_n = 0$  and  $c_n = 2$  ( $b_n = 2$  and  $c_n = 0$ ) if  $a_n = 1$  and  $\mu(n) = 0$  ( $\mu(n) = 2$ ). An interesting case is  $k = 1$ , that is,  $h = 0.1111\dots$ . In this case, if  $1 = x + y$  is the decomposition determined by the choice of some sequence  $\mu$ , then one has precisely  $x = \sum_{n=1}^{\infty} \mu(n)/3^n$ .

*Remark.* The construction of the sequence  $\{L_n\}_n$  (fig. 1-3) is similar to the ones which define by induction the continuous nowhere-differentiable function on  $[0, 1]$  or an infinite homogeneous tree with finite degree. They all provide examples of those geometric objects which are nowadays called fractals. A fractal has the property that each of its portions looks exactly like a reduced copy of the whole thing. This "homogeneous" property has often an algebraic counterpart: in the case of the Cantor ternary set, the  $N$ -th step of its geometric construction corresponds to the fact that



every number of the form  $\sum_1^{N+1} a_n/3^n$ ,  $a_n \in \{0, 1, 2\}$  is obtained from the number  $\sum_1^N a_n/3^n$  by making a choice between  $a_{n+1} = 0$ ,  $a_{n+1} = 1$  and  $a_{n+1} = 2$ . The crucial point is that the nature of this choice does not depend on the number and does not depend on  $N$ . In  $F_n$ , the free group with  $n$  generators, the choice that one makes to form a word of length  $N + 1$  from a word of length  $N$  is independent of either the word or  $N$ . Accordingly, the graph of  $F_n$  is a homogeneous tree (of degree  $2n$ ).

### CANTOR SETS OF CONTINUED FRACTIONS

Cantor point sets play an important role in measure theory and in the theory of continued fractions. The Cantor ternary set  $C$  is a basic example of an uncountable Borel-measurable set whose measure is zero (see, for example, [5], p. 44 and 63). An important object in the theory of continued fractions is the set  $F(n) = \{x \in [0, 1] : x = [0; a_1, a_2, a_3, \dots]$  and  $a_i \leq n$  for all  $i\}$ , that is, the set of continued fractions of bound  $n$  ( $n$  being any positive integer). The fact that  $F(n)$  is a Cantor point set depends on the property that if

$$x = [0; a_1, \dots, a_m, b_{m+1}, b_{m+2}, \dots] \quad \text{and} \quad y = [0; a_1, \dots, a_m, c_{m+1}, c_{m+2}, \dots]$$

are in  $F(n)$ , then  $x < y$  ( $x > y$ ) if  $b_{m+1} < c_{m+1}$  and  $m$  is odd ( $m$  is even). In particular,

$$\min F(n) = [0; n, 1, n, 1, \dots], \quad \max F(n) = [0; 1, n, 1, n, \dots]$$

and  $F(n)$  can be obtained by first removing from  $(0, 1)$  the open intervals

$$(0, [0; n, 1, n, 1, \dots]) \quad \text{and} \quad ([0; 1, n, 1, n, \dots], 1),$$

then removing the intervals

$$\begin{aligned} &([0; n, n, 1, n, 1, \dots], [0; n-1, 1, n, 1, n, \dots]), \\ &([0; n-1, n, 1, n, 1, \dots], [0; n-2, 1, n, 1, n, \dots]), \\ &\dots, ([0; 2, n, 1, n, 1, \dots], [0; 1, 1, n, 1, n, \dots]), \end{aligned}$$

and so on (see [3], p. 971).

A theorem of M. Hall Jr. says that  $F(4) + F(4) + \mathbf{Z} = \mathbf{R}$  ([3], theorem 3.1), which is the analogue of  $C + C = [0, 2]$ . Hall actually proves more general theorems on the nature of  $L(A) + L(B)$  for arbitrary Cantor point sets  $L(A)$  and  $L(B)$ . One of the main applications of Hall's theorem is the result