§2. Type 1 case

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In this section we consider the case where a meridian of L^n in M^{n+2} has infinite order in $H_1(M-L;\mathbf{Z})$. We shall denote by $[m]$ the homology class in $H_1(M-L;\mathbf{Z})$ represented by a meridian m of L in M. For a manifold pair (X, Y) of codimension 2 and an epimorphism γ from $\pi_1(X-Y)$ to a finite group, let $(X, Y)_\gamma$ be the branched covering of (X, Y) corresponding to γ. Each knot group $\pi_1(S^{n+2}-K)$ has a natural epimorphism to $\mathbf{Z}_p$ for any positive integer p, and the corresponding p-fold branched cyclic covering of (S^{n+2}, K) is denoted by $(S^{n+2}, K)_p$.

Lemma 2.1. *Suppose* $[m]$ *is of infinite order. Then if* $(S^{n+2}, K) \in I(M, L)$ *then* $(S^{n+2}, K)_p$ *is a homotopy* $(n+2)$*-sphere for any positive integer* p.

Proof. Since $[m]$ represents a nontrivial element in the finitely generated free abelian group $B_1(M-L) \equiv H_1(M-L;\mathbf{Z})/\mathrm{Tor}\, H_1(M-L;\mathbf{Z})$, there is a positive integer r and a primitive element x in $B_1(M-L)$ such that $[m] = rx$ in $B_1(M-L)$. For each positive integer p, ler γ_p be the canonical epimorphism $\pi_1(M-L) \to B_1(M-L) \otimes \mathbf{Z}_{pr}$. Noting the naturality of the homomorphism γ_p, we can see the following:

$$\begin{aligned}(M, L)_{\gamma_p} &= \big((M, L) \# (S^{n+2}, K)\big)_{\gamma_p \circ f_*} \\ &= (M, L)_{\gamma_p} \# \ d_p (S^{n+2}, K)_p\end{aligned}$$

Here f is a diffeomorphism $(M, L) \# (S^{n+2}, K) \to (M, L)$ and d_p is the order of $B_1(M-L) \otimes \mathbf{Z}_{pr}$ divided by p. Hence $H_*((S^{n+2}, K)_p;\mathbf{Z}) \simeq H_*(S^{n+2};\mathbf{Z})$ and $\pi_1((S^{n+2}, K)_p) \simeq 1$ by the existence of prime decompositions of finitely generated groups into free products [Wg]. Q.E.D.

It is conjectured that those knots which satisfy the conclusion of the above lemma are trivial. In fact, for $n = 1$, it follows from the Smith conjecture [MB]. As a supporting evidence for higher dimensional cases, we have

Lemma. *Suppose that* $(S^{n+2}, K)_p$ *is a homology* $(n+2)$*-sphere for every positive integer* p. *Then the Alexander modules of* K *are trivial.*

Proof. Let $\tilde{E}(K)$ be the infinite cyclic cover of the exterior $E(K)$ of K in S^{n+2}, and let t denote the automorphism of the homology group of $\tilde{E}(K)$ induced by the action of a meridian. Then, by the arguments of [Sm1], we can see that $t^p - 1: H_q(\tilde{E}(K); \mathbf{Z}_r) \to H_q(\tilde{E}(K); \mathbf{Z}_r)$ is an isomorphism for any positive integers p, q, and r. Assume r is prime. Then $H_q(\tilde{E}(K); \mathbf{Z}_r)$ is a finite abelian group, since it is a finitely generated torsion module over the principal ideal domain $\mathbf{Z}_r<t>$ (see [Le3, p. 8]). So the automorphism t on $H_q(\tilde{E}(K); \mathbf{Z}_r)$ has a finite order, say d, and we have $t^d - 1 = 0$. Hence $H_q(\tilde{E}(K); \mathbf{Z}_r) = 0$, and by the universal coefficient theorem, the following holds for any prime r and any positive integer q:

$$H_q(\tilde{E}(K); \mathbf{Z}) \otimes \mathbf{Z}_r = 0 \tag{2.3}$$

$$\operatorname{Tor}(H_q(\tilde{E}(K); \mathbf{Z}), \mathbf{Z}_r) = 0 \tag{2.4}$$

By (2.4), $H_q(\tilde{E}(K); \mathbf{Z})$ has no nontrivial elements of finite order; so it has a square presentation matrix $M(t)$ as a $Z<t>$-module by [Le3, Proposition 3.5]. By (2.3) the q-th Alexander polynomial $\det M_q(t)$ $(\in \mathbf{Z}<t>)$ is a unit mod. r for any prime r. Hence it is a unit in $\mathbf{Z}<t>$, and we have $H_q(\tilde{E}(K); \mathbf{Z}) = 0$ for any positive integer q. Q.E.D.

Thus, as a consequence of Lemmas 2.1 and 2.2 and the results of [Le2] and [T], we have the following:

PROPOSITION 2.5. *Suppose* $[m]$ *is of infinite order. Then any knot in* $I(M, L)$ *has trivial Alexander modules and is null cobordant.*

Hence the only obstruction for a knot (S^{n+2}, K) in $I(M, L)$ to be trivial lies in the knot group $\pi_1(S^{n+2} - K)$. For the special case where $[m]$ generates $H_1(M - L)$, we can apply the result of Maeda [Ma] (cf. [DF]), and obtain the following:

THEOREM 2.6. *Suppose* $n \geqslant 3$ *and* $H_1(M - L)$ *is the infinite cyclic group generated by* $[m]$. *Then* $I(M, L)$ *is trivial.*

Proof. Let (S^{n+2}, K) be a knot in $I(M, L)$. Note that $\pi_1(M - L)$ is isomorphic to the amalgamated free product $\pi_1(M - L) \underset{<m>}{*} \pi_1(S^{n+2} - K)$. Then we can conclude $\pi_1(S^{n+2} - K) \simeq \mathbf{Z}$ by the result of [Ma] (cf. [DF]) which asserts the existence of a prime decomposition of a finitely presented group G with $G/[G, G] \simeq \mathbf{Z}$ with respect to such amalgamated free products. Combined with Proposition 2.5, we see $S^{n+2} - K$ is homotopy equivalent to a circle. Hence (S^{n+2}, K) is trivial by [Le1].