

## 4. Relative de Rham homology

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If we combine theorem (3.2) and the biduality theorem (2.1) we obtain what is usually known as the

(3.6) DE RHAM THEOREM. *Integration over smooth singular simplexes induces an isomorphism*

$$H^\bullet(X, \mathbf{C}) \xrightarrow{\sim} H_\infty^\bullet(X, \mathbf{C})$$

*from de Rham cohomology to smooth singular cohomology.*

#### 4. RELATIVE DE RHAM HOMOLOGY

Let us start by some general remarks on the support of a compact  $p$ -chain  $T$  on a smooth  $n$ -dimensional manifold  $X$ . Since we can realize  $T$  as a section in the sheaf  $\Omega^p^\vee$  the general sheaf theoretic notion of support applies: The *support* of  $T$ ,  $\text{Supp}(T)$  is the smallest closed subset  $Z$  of  $X$ , such that the restriction of  $T$  to  $X - Z$  is zero.

(4.1) EXAMPLE. Integration over an oriented compact  $p$ -dimensional submanifold  $K$  with boundary defines a compact  $p$ -chain  $\kappa$  with  $\text{Supp}(\kappa) = K$ . From Stokes formula

$$\int_K d\omega = \int_{\partial K} \omega, \quad \omega \in \Gamma(X, \Omega^p),$$

we conclude that  $\text{Supp}(b\kappa) = \partial K$ .

Let us now consider the inclusion  $j: U \rightarrow X$  of an open subset  $U$  of  $X$ . The induced map

$$j_*: D_p^c(U, \mathbf{C}) \rightarrow D_p^c(X, \mathbf{C}), \quad p \in \mathbf{N},$$

is injective since we may interpret  $j_*$  as "extension by zero" in the sheaf  $\Omega_p^\vee$ , compare (2.5). A compact  $p$ -chain  $T$  on  $X$  belongs to the image of  $j_*$  if and only if  $\text{Supp}(T) \subseteq U$ . The complex  $D_p^c(X, U; \mathbf{C})$  of *relative compact chains* is defined to fit into the following exact sequence

$$(4.2) \quad 0 \rightarrow D_p^c(U, \mathbf{C}) \xrightarrow{j_*} D_p^c(X, \mathbf{C}) \rightarrow D_p^c(X, U; \mathbf{C}) \rightarrow 0.$$

On this basis we can define the *relative de Rham* homology group

$$H_p(X, U; \mathbf{C}) = H_p D_p^c(X, U; \mathbf{C}), \quad p \in \mathbf{N}.$$

In more concrete terms we can describe this homology group as

$$(4.3) \quad \{Z \in D_p^c(X, \mathbf{C}) \mid \text{Supp}(bZ) \subseteq U\} \Big/ \begin{array}{l} \{bW \mid W \in D_{p+1}^c(X, \mathbf{C})\} \\ + \{Z \in D_p(X, \mathbf{C}) \mid \text{Supp}(Z) \subseteq U\} \end{array}$$

From the exact sequence (4.2) we deduce the homology sequence

$$(4.4) \quad \begin{array}{l} \rightarrow H_p^c(U, \mathbf{C}) \rightarrow H_p^c(X, \mathbf{C}) \rightarrow H_p^c(X, U; \mathbf{C}) \\ \rightarrow H_{p-1}^c(U, \mathbf{C}) \rightarrow H_{p-1}^c(X, \mathbf{C}) \rightarrow \end{array}$$

Let  $f: X \rightarrow Y$  denote a smooth map,  $U$  an open subset of  $X$  and  $V$  an open subset of  $Y$  containing  $f(U)$ . Let us notice that

$$(4.5) \quad \text{Supp}(f_*T) \subseteq f(\text{Supp}(T)), \quad T \in D_p^c(X, \mathbf{C}).$$

These remarks make it evident, that de Rham homology is a covariant functor on the category of pairs consisting of a manifold and one of its open subspaces.

(4.6) *Excision.* Let  $Z$  be a closed subset of  $X$  and  $Y$  an open subset of  $X$  containing  $Z$ . The inclusion of  $V = Y - Z$  in  $U = X - Z$  induces an isomorphism

$$H^c(Y, V; \mathbf{C}) \xrightarrow{\sim} H^c(X, U; \mathbf{C}).$$

*Proof.* Let  $i: Z \rightarrow X$  denote the inclusion. From the fact that  $\Omega^{\cdot \vee}$  consists of soft sheaves we deduce an exact sequence

$$0 \rightarrow \Gamma_c(U, \Omega^{\cdot \vee}) \rightarrow \Gamma_c(X, \Omega^{\cdot \vee}) \rightarrow \Gamma_c(Z, i^*\Omega^{\cdot \vee}) \rightarrow 0$$

compare [5] III. 7.6. This allows us to make the identification

$$(4.7) \quad D^c(X, U; \mathbf{C}) \xrightarrow{\sim} \Gamma_c(Z, i^*\Omega^{\cdot \vee}), \quad Z = X - U.$$

The expression on the right hand side is unchanged, when  $X$  is replaced by  $Y$  and  $U$  by  $V$ . Q.E.D.

(4.8) *Continuity.* Let  $(X_\alpha)$  be an outward directed open covering of the manifold  $X$ . For any open subset  $U$  of  $X$  we have that

$$\lim_{\rightarrow \alpha} H^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) = H^c(X, U; \mathbf{C})$$

*Proof.* As a consequence of the theorem of Borel-Heine, see possibly [5] III. 5.2, we find that

$$\lim_{\rightarrow} D^c(X_\alpha, \mathbf{C}) = D^c(X, \mathbf{C})$$

and similarly with  $X$  replaced by  $U$  and  $X_\alpha$  replaced by  $U \cap X_\alpha$ . Using this and the exact sequence 4.2 we get that

$$\lim_{\rightarrow} D_c^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) = D_c^c(X, U; \mathbf{C})$$

from which the result follows by passing to homology. Q.E.D.

Let us also notice that in case  $X$  is the disjoint union of a family  $(X_\alpha)$  of open subsets we have that

$$(4.9) \quad \bigoplus_{\alpha} H_c^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) \xrightarrow{\sim} H_c^c(X, U; \mathbf{C}).$$

### 5. STOKES FORMULA

Let us consider the open subset  $U$  of the  $n$ -dimensional smooth manifold  $X$  and the resulting exact sequences

$$(5.1) \quad \begin{aligned} &\rightarrow H_p^c(X, \mathbf{C}) \rightarrow H_p^c(X, U; \mathbf{C}) \xrightarrow{b} H_{p-1}^c(U, \mathbf{C}) \xrightarrow{j^*} H_{p-1}^c(X, \mathbf{C}) \rightarrow \\ &\leftarrow H^p(X, \mathbf{C}) \leftarrow H^p(X, U; \mathbf{C}) \xleftarrow{\partial} H^{p-1}(U, \mathbf{C}) \xleftarrow{j^*} H^{p-1}(X, \mathbf{C}) \leftarrow \end{aligned}$$

where the first is discussed in the previous section and the second is the sheaf cohomology sequence. The relative term in the second sequence is often written

$$(5.2) \quad H_Z^p(X, \mathbf{C}) = H^p(X, U; \mathbf{C}), \quad Z = X - U.$$

We can now extend the biduality theorem (2.1).

(5.3) THEOREM. *The cohomology sequence above is dual to the homology sequence. In particular we have a Stoke's formula*

$$\langle b\alpha, \omega \rangle = \langle \alpha, \partial\omega \rangle$$

for  $\alpha \in H_p^c(X, U; \mathbf{C})$  and  $\omega \in H^{p-1}(U, \mathbf{C})$ .

*Proof.* The first sequence arises from the following short exact sequence of complexes, compare (4.2) and (4.7),

$$0 \rightarrow \Gamma_c(U, \Omega^{\bullet \vee}) \xrightarrow{j^*} \Gamma_c(X, \Omega^{\bullet \vee}) \rightarrow \Gamma_c(Z, \Omega^{\bullet \vee}) \rightarrow 0.$$

In order to calculate the second sequence we depart from the flabby resolution  $\Omega^{\bullet \vee \vee}$  of  $\mathbf{R}$  established in the proof of the biduality theorem (2.1).