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AN ALGEBRAIC PROOF OF VAN DER WAERDEN'S THEOREM

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0. INTRODUCTION. In 1927 van der Waerden [9] proved that if the natural numbers **N** are partitioned into finitely many classes in any way whatever, one of these classes contains arbitrarily long arithmetic progressions. Since then several different proofs have been given, most of them of a combinatorial nature ([5], [7], and [8]), and one proof [4] based on ideas of dynamics. We present here yet another proof whose basic idea is algebraic and which makes essential use of the Stone-Čech compactification of the natural numbers. While the proof is not elementary it is a conceptual proof which presents in a simple context algebraic and topological machinery which is useful in dealing with a variety of combinatorial questions. In particular it confirms that the Stone-Čech compactification which has shown up in a number of other questions in Ramsey Theory plays a significant role also for one of the best known results in Ramsey Theory — van der Waerden's Theorem.

To underscore the algebraic ingredients of the proof, let us state a theorem regarding finite semigroups which embodies the kernel of our argument. So let S be a finite semigroup whose operation will be denoted by multiplication. We say J is a *(two-sided) ideal* in S if $SJ \subseteq J$ and $JS \subseteq J$. If S is a semigroup denote by S^l the l-fold cartesian product of S with itself. S^l_Δ will denote the diagonal of S^l, i.e. $S^l_\Delta = \{(x, x, ..., x) : x \in S\}$. Then S^l_Δ and S^l are themselves semigroups with the coordinatewise operations.

0.1 THEOREM. *Assume S is a finite semigroup. If R is a semigroup in S^l with $S^l_\Delta \subseteq R$ and if J is any two-sided ideal of R, then J must meet S^l_Δ.*

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In general this result does not hold for infinite semigroups. We invite the reader to find a counterexample for $S = (\mathbf{Z}, +)$. However it does hold for compact semigroups as we will show implicitly in the proof of Theorem 3.3. (Of course the finite version is then a special case.)

We intend to apply this theorem to the natural numbers $\mathbf{N}$ by compactifying $\mathbf{N}$ in such a way so as to obtain a compact semigroup; this is the role of the Stone-Čech compactification $\beta\mathbf{N}$ of $\mathbf{N}$. We obtain a theorem about $\beta\mathbf{N}$ which when unraveled becomes exactly van der Waerden's Theorem.

We warn the reader that in the compactification of $\mathbf{N}$ the operation of addition will be extended with the usual notation $+$. However the semigroup will not be commutative and so one has to accustom oneself to non-commutative addition.

1. Semigroup properties of $\beta\mathbf{N}$

Any completely regular Hausdorff space has a maximal compactification, the Stone-Čech compactification. In particular the discrete space $\mathbf{N}$ of positive integers has a Stone-Čech compactification $\beta\mathbf{N}$ which is characterized by: (1) $\beta\mathbf{N}$ is a compact Hausdorff space; (2) $\mathbf{N}$ is a dense subset of $\beta\mathbf{N}$; and (3) given any compact Hausdorff space Y and any $f : \mathbf{N} \to Y$ there is a continuous extension $f^\beta : \beta\mathbf{N} \to Y$, (that is $f^\beta|_{\mathbf{N}} = f$).

Our proof of van der Waerden's Theorem is based on the fact that the operation of ordinary addition extends to $\beta\mathbf{N}$ as an operation which we denote by $+$. $\beta\mathbf{N}$ under this operation will be a semigroup in which the operation of addition is continuous in a restricted way. Namely let $(S, +)$ be a semigroup with S a topological space and define functions ρ_x and λ_x for each $x \in S$ by $\rho_x(y) = y + x$ and $\lambda_x(y) = x + y$. If one requires only that ρ_x be continuous, S is called a right topological semigroup.

1.1 Lemma. *There is an operation* $+$ *on* $\beta\mathbf{N}$ *such that* $\beta\mathbf{N}$ *is a compact right topological semigroup,* $+$ *extends ordinary addition on* $\mathbf{N}$, *and* λ_n *is continuous for each* $n \in \mathbf{N}$.

Proof. We extend $+$ in stages, starting with $+$ defined on $\mathbf{N} \times \mathbf{N}$. Given $n \in \mathbf{N}$, consider $f_n : \mathbf{N} \to \beta\mathbf{N}$ defined by $f_n(m) = n + m$. Then each f_n has a continuous extension $f_n^\beta : \beta\mathbf{N} \to \beta\mathbf{N}$. For $n \in \mathbf{N}$ and $p \in \beta\mathbf{N}\backslash\mathbf{N}$ define

$n + p = f_n^\beta(p)$. (Then for $n \in \mathbf{N}$ and any $p \in \beta\mathbf{N}$, $n + p = f_n^\beta(p)$ since if $p \in \mathbf{N}$, $f_n^\beta(p) = f_n(p) = n + p$.) Now $+$ is defined on $\mathbf{N} \times \beta\mathbf{N}$. Given $p \in \beta\mathbf{N}$ define $g_p : \mathbf{N} \to \beta\mathbf{N}$ by $g_p(n) = n + p$. Then each g_p has a continuous extension $g_p^\beta : \beta\mathbf{N} \to \beta\mathbf{N}$. Then for $p \in \beta\mathbf{N}$ and $q \in \beta\mathbf{N}\backslash\mathbf{N}$ define $q + p = g_p^\beta(q)$. (Again if p, q are any points in $\beta\mathbf{N}$ we have $q + p = g_p^\beta(q)$.)

Since for any $n \in \mathbf{N}$, $\lambda_n = f_n^\beta$ and for any $p \in \beta\mathbf{N}$, $\rho_p = g_p^\beta$, the continuity assumptions are immediate. Thus we need only check that the operation is associative. To this end let $p, q, r \in \beta\mathbf{N}$. Observe that $p + (q+r) = \rho_{q+r}(p)$ while $(p+q) + r = (\rho_r \circ \rho_q)(p)$ so by continuity it suffices to show ρ_{q+r} and $\rho_r \circ \rho_q$ agree on the dense subset $\mathbf{N}$ of $\beta\mathbf{N}$. Let $n \in \mathbf{N}$. Then

$$\rho_{q+r}(n) = n + (q+r) = (\lambda_n \circ \rho_r)(q)$$

$$\text{and} \quad (\rho_r \circ \rho_q)(n) = (n+q) + r = (\rho_r \circ \lambda_n)(q).$$

Again by continuity, it suffices to show $\lambda_n \circ \rho_r$ and $\rho_r \circ \lambda_n$ agree on $\mathbf{N}$. Let $m \in \mathbf{N}$. Then

$$(\lambda_n \circ \rho_r)(m) = n + (m+r) = (\lambda_n \circ \lambda_m)(r)$$

while

$$(\rho_r \circ \lambda_n)(m) = (n+m) + r = \lambda_{n+m}(r).$$

Thus it finally suffices to show $\lambda_n \circ \lambda_m$ and λ_{n+m} agree on $\mathbf{N}$. Let $t \in \mathbf{N}$. Then $(\lambda_n \circ \lambda_m)(t) = n + (m+t) = (n+m) + t = \lambda_{n+m}(t)$ as required. □

The main fact about $\beta\mathbf{N}$ making it useful for van der Waerden's Theorem and similar results is the content of the following lemma.

1.2 LEMMA. *If* $\{A_1, A_2, ..., A_m\}$ *is a finite partition of* $\mathbf{N}$, *then* $\{cl\, A_1, cl\, A_2, ..., cl\, A_m\}$ *is a partition of* $\beta\mathbf{N}$ *such that for each* $i \in \{1, 2, ..., m\}$, $cl\, A_i$ *is open.*

Proof. Let $Y = \{1, 2, ..., m\}$ with the discrete topology and define $f : \mathbf{N} \to Y$ by $f(n) = i$ if and only if $n \in A_i$. For each $i \in \{1, 2, ..., m\}$, let $B_i = \{p \in \beta\mathbf{N} : f^\beta(p) = i\}$. Then immediately $\{B_1, B_2, ..., B_m\}$ is a partition of $\beta\mathbf{N}$. Further, given $i \in \{1, 2, ..., m\}$, $B_i = (f^\beta)^{-1}[\{i\}]$. Since $\{i\}$ is open and closed in Y and f^β is continuous, B_i is open and closed. Since $A_i \subseteq B_i$, one has $cl\, A_i \subseteq B_i$. To see that $B_i \subseteq cl\, A_i$, let $x \in B_i$ and let U be a neighborhood of x. Since $\mathbf{X}$ is dense in $\beta\mathbf{N}$, pick $y \in \mathbf{N} \cap (U \cap B_i)$. Since $y \in B_i$, $f(y) = i$ so $y \in A_i$. Thus $U \cap A_i \neq \emptyset$ as required. □

2. Minimal left ideals in right topological semigroups

We present in this section several well known facts which are not usually seen in early graduate courses.

2.1 Lemma (Ellis [2]). *Let S be a compact Hausdorff right topological semigroup. Then S has an idempotent, that is there exists $x \in S$ with $x + x = x$.*

Proof. Let $\mathscr{A} = \{A \subseteq S: A \neq \emptyset, A \text{ is compact, and } A + A \subseteq A\}$. Now $\mathscr{A} \neq \emptyset$ since $S \in \mathscr{A}$. Let $\mathscr{C}$ be a chain in A. Then $\mathscr{C}$ is a collection of closed subsets of S with the finite intersection property so $\cap\mathscr{C} \neq \emptyset$ and $\cap\mathscr{C}$ is compact. Trivially $(\cap\mathscr{C}) + (\cap\mathscr{C}) \subseteq \cap\mathscr{C}$. Pick by Zorn's Lemma a minimal member A of $\mathscr{A}$.

Pick $x \in A$ and let $B = A + x$. Now $B = \rho_x[A]\ (=\{\rho_x(y): y \in A\})$ so, as the continuous image of a compact set, B is compact (and trivially non-empty). Also $B + B = A + x + A + x \subseteq A + A + A + x \subseteq A + x = B$. Thus $B \in \mathscr{A}$. Since $B = A + x \subseteq A + A \subseteq A$ and A is minimal, $B = A$ so $x \in B = A + x$. That is, there exists $y \in A$ with $x = y + x$.

Let $C = \{y \in A: x = y + x\}$. ρ_x is continuous so $\rho_x^{-1}[\{x\}]$ is closed. Thus C is closed, hence compact. To see that $C + C \subseteq C$, let $y, z \in C$. Then $y + z \in A$ and $(y+z) + x = y + (z+x) = y + x = x$ so $y + z \in C$. Thus $C \in \mathscr{A}$. Since $C \subseteq A$ and A is minimal, $C = A$. Then $x \in C$ and hence $x + x = x$. □

A non-empty subset I of a semigroup S is a left ideal if $S + I \subseteq I$, a right ideal if $I + S \subseteq I$, and a two-sided ideal if it is both a left ideal and a right ideal. It is a fact (which we will not need) that any right ideal in a compact right topological semigroup contains a minimal right ideal, which need not be closed. (For this and other interesting facts see [1].) We do need a similar fact about left ideals.

2.2 Lemma. *Let S be a compact Hausdorff right topological semigroup. Any left ideal contains a minimal left ideal and minimal left ideals are closed.*

Proof. Let L be a left ideal of S. Let $\mathscr{A} = \{A \subseteq L: A \text{ is a left ideal and } A \text{ is compact}\}$. Choose $x \in L$. Then $S + x = \rho_x[S]$, the continuous image of a compact space. Also $S + (S+x) = (S+S) + x \subseteq S + x$ so $S + x$ is a left ideal. Since $S + x \subseteq S + L \subseteq L$, we have $\mathscr{A} \neq \emptyset$. One easily sees that the intersection of a chain in $\mathscr{A}$ is again in $\mathscr{A}$. Choose by Zorn's Lemma a minimal member A of $\mathscr{A}$.

To see that A is in fact a minimal left ideal, assume we have a left ideal $B \subseteq A$ and pick $x \in B$. Then as above $S + x \in \mathscr{A}$ while $S + x \subseteq S + B \subseteq B \subseteq A$ so $S + x = A$ so $B = A$ $\square$

2.3 *Definition.* Let S be a semigroup. Then $M(S) = \cup \{L: L$ is a minimal left ideal of $S\}$.

It is a fact (which we will not need) that if S is a compact Hausdorff right topological semigroup, then $M(S)$ is a two-sided ideal of S.

2.4 LEMMA. *Let S be a compact Hausdorff right topological semigroup and let I be a two-sided ideal of S. Then $M(S) \neq \emptyset$ and $M(S) \subseteq I$.*

Proof. Since S is a left ideal of S it contains by Lemma 2.2 a minimal left ideal so $M(S) \neq \emptyset$. So see that $M(S) \subseteq I$, let $x \in M(S)$. There is a minimal left ideal L of S with $x \in L$. Also choose some $y \in I$. Then $y + x \in L \cap I$ (since I is a right ideal) so $L \cap I \neq \emptyset$. Thus $L \cap I$ is a left ideal contained in L so that $L \cap I = L$. $\square$

The proof of the following lemma is an easy exercise and we omit it.

2.5 LEMMA. *Let S_1 and S_2 be compact right topological semigroups and let $S_1 \times S_2$ have the product topology and coordinatewise operations. Then $S_1 \times S_2$ is a compact right topological semigroup. Given $x \in S_1$ and $y \in S_2$, λ_x and λ_y may or may not be continuous (where $\lambda_x(t) = x + t$). If $\lambda_x: S_1 \to S_1$ and $\lambda_y: S_2 \to S_2$ are continuous, then $\lambda_{(x,y)}: S_1 \times S_2 \to S_1 \times S_2$ is continuous.*

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We let $l \in \mathbf{N}$ be fixed throughout and show that given any finite partition of $\mathbf{N}$ some one cell contains a length l arithmetic progression.

3.1 *Definition.* (a) Let $Y = (\beta\mathbf{N})^l$ with the product topology and coordinatewise operations.

(b) $E^* = \{(a, a + d, a + 2d, \ldots, a + (l-1)d): a \in \mathbf{N}$ and $d \in \mathbf{N} \cup \{0\}\}$.

(c) $I^* = \{(a, a + d, a + 2d, \ldots, a + (l-1)d): a, d \in \mathbf{N}\}$.

(d) $E = cl_Y E^*$

(e) $I = cl_Y I^*$.

Note that by Lemmas 1.1 and 2.5, Y is a compact Hausdorff right topological semigroup and whenever $\mathbf{x} = (x_1, x_2, \ldots, x_l) \in \mathbf{N}^l$, $\lambda_{\mathbf{x}}$ is continuous.

3.2 LEMMA. *E is a compact Hausdorff right topological semigroup and I is a two sided ideal of E.*

Proof. Compactness is immediate and the Hausdorff property and right continuity are inherited from Y. We let $\mathbf{p} = (p_1, p_2, \dots p_l)$ and $\mathbf{q} = (q_1, q_2, \dots, q_l)$ be members of E and show that $\mathbf{p} + \mathbf{q} \in E$. We show further that if either $\mathbf{p}$ or $\mathbf{q}$ is in I, then $\mathbf{p} + \mathbf{q} \in I$.

To see that $\mathbf{p} + \mathbf{q} \in E$, let U be a neighborhood of $\mathbf{p} + \mathbf{q}$. By the continuity of $\rho_\mathbf{q}$, pick a neighborhood V of $\mathbf{p}$ with $V + \mathbf{q} = \rho_\mathbf{q}[V] \subseteq U$. Since $\mathbf{p} \in cl\, E^*$ we may pick $a \in \mathbf{N}$ and $d \in \mathbf{N} \cup \{0\}$ with

$$(a, a+d, a+2d, \dots, a+(l-1)d) \in V.$$

If $\mathbf{p} \in I$ we may presume $d \neq 0$. Let $\mathbf{x} = (a, a+d, a+2d, \dots, a+(l-1)d)$. Then $\mathbf{x} \in V$ so $\mathbf{x} + \mathbf{q} \in U$. By the continuity of $\lambda_\mathbf{x}$, pick a neighborhood W of $\mathbf{q}$ with $\mathbf{x} + W = \lambda_\mathbf{x}[W] \subseteq U$. Since $\mathbf{q} \in cl\, E^*$, pick $b \in \mathbf{N}$ and $c \in \mathbf{N} \cup \{0\}$ (with $c \neq 0$ if $\mathbf{q} \in I$) such that $(b, b+c, b+2c, \dots, b+(l-1)c) \in W$. Let $\mathbf{y} = (b, b+c, b+2c, \dots, b+(l-1)c)$. Then $\mathbf{x} + \mathbf{y} \in U \cap E^*$. If either $d \neq 0$ or $c \neq 0$, then $c + d \neq 0$ so $\mathbf{x} + \mathbf{y} \in U \cap I^*$. □

3.3 THEOREM. *Let $p \in M(\beta\mathbf{N})$ and let $\mathbf{p} = (p, p, \dots, p)$. Then $\mathbf{p} \in I$.*

Proof. We first show that $\mathbf{p} \in E$. Let $U_1 \times U_2 \times \dots \times U_l$ be a basic neighborhood of $\mathbf{p}$. Then $U_1 \cap U_2 \cap \dots \cap U_l$ is a neighborhood of p in $\beta\mathbf{N}$. Since $\mathbf{N}$ is dense, pick $a \in \mathbf{N} \cap (U_1 \cap U_2 \cap \dots \cap U_l)$. Then $(a, a, \dots, a) \in E^* \cap (U_1 \times U_2 \times \dots \times U_l)$. Thus $\mathbf{p} \in cl\, E^* = E$.

Since $p \in M(\beta\mathbf{N})$, there is a minimal left ideal L of $\beta\mathbf{N}$ with $p \in L$. Since $E + \mathbf{p}$ is a left ideal of E, pick by Lemma 2.2 a minimal left ideal L^* of E with $L^* \subset E + \mathbf{p}$. Since L^* is closed, hence compact, pick by Lemma 2.1 an idempotent $\mathbf{q} = (q_1, q_2, \dots, q_l)$ in L^*. Now $\mathbf{q} \in L^* \subseteq E + \mathbf{p}$ so pick some $\mathbf{s} = (s_1, s_2, \dots, s_l)$ in E with $\mathbf{q} = \mathbf{s} + \mathbf{p}$.

We show that $\mathbf{p} + \mathbf{q} = \mathbf{p}$. To this end let $i \in \{1, 2, \dots, l\}$. Now $q_i = s_i + p \in L$ so $\beta\mathbf{N} + q_i \subseteq \beta\mathbf{N} + L \subseteq L$. Thus $\beta\mathbf{N} + q_i$ is a left ideal contained in the minimal left ideal L so that $\beta\mathbf{N} + q_i = L$. Thus since $p \in L$ there exists $t_i \in \beta\mathbf{N}$ with $t_i + q_i = p$. But then $p + q_i = t_i + q_i + q_i = t_i + q_i = p$ as required.

Since $\mathbf{p} \in E$ and $\mathbf{q} \in L^*$, a left ideal of E, we have $\mathbf{p} = \mathbf{p} + \mathbf{q} \in L^*$ so that $\mathbf{p} \in M(E)$. Thus by Lemma 2.4, $\mathbf{p} \in I$.

3.4 COROLLARY (van der Waerden). *Let $m \in \mathbf{N}$ and let $\{A_1, A_2, \dots, A_m\}$ be a partition of $\mathbf{N}$. There exist $i \in \{1, 2, \dots, m\}$ and $a, d \in \mathbf{N}$ with $\{a, a + d, a + 2d, \dots, a + (l-1)d\} \subseteq A_i$.*

Proof. By Lemma 2.4 $M(\beta\mathbf{N}) \neq \emptyset$ so pick $p \in M(\beta\mathbf{N})$ and let $\mathbf{p} = (p, p, \ldots, p)$. By Lemma 1.2 pick $i \in \{1, 2, \ldots, m\}$ such that $cl\, A_i$ is a neighborhood of p and let $U = cl\, A_i$. Then $U \times U \times \ldots \times U$ is a neighborhood of $\mathbf{p}$ while, by Theorem 2.3, $\mathbf{p} \in I = cl\, I^*$. Pick $a, d \in \mathbf{N}$ with $(a, a+d, a+2d, \ldots, a+(l-1)d) \in U \times U \times \ldots \times U$. Then

$$\{a, a + d, a + 2d, \ldots, a + (l-1)d\} \subseteq U \cap \mathbf{N} = (cl\, A_i) \cap N = A_i . \qquad \square$$

We remark that if one starts with the free semigroup on l letters in place of $\mathbf{N}$, essentially the same proof yields the Hales-Jewett Theorem. See [3] for the details.

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