III. Classification

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(h) Nonions (1882)

Sylvester wrote many papers between 1882 and 1884 on hypercomplex number systems, matrices, and their connections. (In fact, Sylvester claimed that he had created the theory of matrices independently of Cayley.) One such example was his system of "nonions"—a 9-dimensional algebra over **R**, generated by elements $u^i v^j (i, j = 0, 1, 2)$, where

$$u = \begin{pmatrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0 \end{pmatrix},$$

ρ being a cubic root of 1. He showed that this algebra is isomorphic to the algebra $M_3(\mathbf{R})$ of 3×3 matrices over **R**.

III. Classification

What we have presented above is only a sample, albeit a representative one, of various hypercomplex number systems introduced, largely by British mathematicians, within 30 or 40 years of Hamilton's work on quaternions. The scene now shifted to the United States and to continental Europe. Now that a stock of examples of noncommutative number systems had been established, one could begin to have a theory. The general concept of a (finite-dimensional) associative algebra (a hypercomplex number system) emerged, and there was a move to classify certain types of these general structures. We focus on three such developments.

(a) Low-dimensional algebras

Of fundamental importance here is B. Peirce's groundbreaking paper "Linear Associative Algebra" of 1870. In the last 100 pages of this 150-page paper Peirce classifies algebras (i.e. hypercomplex number systems) of dimension < 6 by giving their multiplication tables. There are, he shows, over 150 such algebras! What is important in this paper, though, is not the classification but the means used to obtain it.[1] For here Peirce introduces concepts, and derives results, which proved fundamental for subsequent developments.

[1]) Algebras of fixed (low) dimension were also classified, using different methods, by Scheffers, Study, and others. See [36], [38], [39], [66] for details. The complexity of the structure of general algebras, even of low dimensions, directed later researchers to focus on the study of special types of algebras (see e.g. (b) and (c) below and sec. IV).

Peirce's work is very much in the abstract Anglo-American tradition (cf. p. 229). Peirce was a great enthusiast of the quaternions and had taught them at Harvard as early as 1848. He was also an adherent of the symbolical approach to algebra. Algebra to Peirce was "formal mathematics", which was mathematics expressed by symbols which were not "tramelled by the conditions of external representation or special interpretation." In fact, Peirce's approach to mathematics in general was abstract, as can be seen from the "definition" of mathematics in the opening sentence of his treatise: "Mathematics is the science which draws necessary conclusions." This was certainly not a prevailing view of mathematics in the 19th century, although it is not unique to Peirce.[1)]

Among the conceptual advances in Peirce's work are:

(1) An "abstract" definition of a *finite-dimensional associative algebra.* Peirce defines such an algebra (he calls it a "linear associative algebra") as the totality of formal expressions of the form $\sum_{i=1}^{n} a_i e_i$, where the e_i are basis elements.[2)] Addition is defined componentwise and multiplication by means of "structural constants" c_{ij}^{k}, namely $e_i e_j = \sum_{k=1}^{n} c_{ij}^{k} e_k$. Associativity under multiplication and distributivity are assumed, but not commutativity. This is probably the earliest conscious and explicit definition of an associative algebra (i.e. a hypercomplex number system).[3)]

(2) The use of *complex coefficients.* Peirce takes the coefficients a_i in the expressions $\sum a_i e_i$ to be *complex* numbers. This conscious broadening of the field of coefficients from **R** to **C** (as we noted, both Hamilton and Clifford

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[1)] Cf. the following "definitions" of mathematics expressing similar sentiments:

Gauss (1831): "Mathematics is concerned only with the enumeration and comparison of relations."

Grassmann (1844): "[Pure] mathematics is the science of forms."

Boole (1847): "It is not the essence of mathematics to be conversant with the ideas of number and quantity."

Hankel (1867): "[Mathematics is] purely intellectual, a pure theory of forms, which has for its objects not the combination of quantities or their images, the numbers, but things of thought to which there could correspond effective objects or relations, even though such a correspondence is not necessary."

Cantor (1883): "Mathematics is entirely free in its development and its concepts are restricted only by the necessity of being noncontradictory."

[2)] Peirce calls the basis elements the "alphabet" of the algebra. An algebra also has a "vocabulary" which consists of the operations of the algebra, as well as a "grammar" which gives the rules of composition (i.e. the postulates).

[3)] De Morgan, in the paper on foundations of algebra which we mentioned above, gave a similar, but less formal, description of such an algebra, calling it "a system of Algebra of the n^{th} character". Moreover, Grassmann in his *Ausdehnungslehre* of 1844, speakes of the "space" of "extensive quantities" (see (c) above).

presented *examples* of algebras with complex coefficients) was an important conceptual advance on the road to coefficients taken from an arbitrary field.

(3) *Identity* for the algebra is *not required.* This, too, is a departure from past practice and, again, gives an indication of Peirce's general, abstract approach. It made the statements and proofs of various results more difficult.

(4) Introduction of *nilpotent and idempotent elements.* An element x of an algebra is nilpotent if $x^n = 0$ for some positive integer n, idempotent if $x^2 = x$. These are two very important concepts which proved basic for the subsequent study of algebras (and later rings). After introducing these concepts [1]) Peirce proved the fundamental result that any algebra contains a nilpotent or an idempotent element. (Recall that the algebra need not have an identity.)

(5) The "Peirce decomposition". Peirce showed that if e is an idempotent of an algebra A then $A = eAe \oplus eB_1 \oplus B_2e \oplus B$, where $B_1 = \{x \in A : xe = 0\}$, $B_2 = \{x \in A : ex = 0\}$, and $B = B_1 \cap B_2$ ($\oplus$ indicates direct sum). This so-called Peirce decomposition of an algebra relative to an idempotent was a fundamental result which enabled Peirce to get a better hold on his algebra by studying its constituent parts. It is a central tool in the study of rings and algebras.

Peirce's work was well ahead of its time, and attracted little attention at first. Cayley, for example, who praised Peirce's work in an address in 1883 to the British Association for the Advancement of Science, called it "outside of ordinary mathematics" [2]). Even some of Peirce's admirers in the United States characterized the work as "philosophy of mathematics" rather than mathematics proper. Peirce, of course, turned out to have been a *mathematical* pioneer. See [62], [66], [69] for details.

(b) DIVISION ALGEBRAS

As we mentioned, the first example of a noncommutative algebra, namely Hamilton's quaternions, was a division algebra. The question arose as to which other systems of n-tuples of real numbers (hypercomplex number systems) possessed unique division (i.e. were division algebras). The answer was given, independently, by Frobenius (in 1878) and by C.S. Peirce (B. Peirce's son, in 1881), namely that the real numbers, the complex numbers, and the quater-

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[1]) Idempotent elements appeared in the work of Boole twenty years earlier.

[2]) This is ironic, coming from Cayley. His own work of 1854 on abstract groups was neglected by the mathematical community for twenty years! See [49].

nions are (in our terminology) the only possible finite-dimensional associative division algebras over **R**.

Frobenius' work appears at the end of a seminal paper entitled "Über lineare substitutionen and bilineare Formen", in which he develops the theory of matrices in the language of bilinear forms. (The forms, he says, can be viewed "as a system of n^2 quantities which are ordered in n rows and n columns.") In the final section of the paper Frobenius defines a hypercomplex number system (he calls it a "form system") as consisting of elements of the form $\sum_{i=1}^{m} a_i E_i (a_i \in \mathbf{R}$ *or* $\mathbf{C})$, where the E_i are *some* linearly independent bilinear forms in the variables $x_1, \ldots, x_n; y_1, \ldots, y_n$ such that the product of any two of them is again a linear combination of $E_1, E_2, \ldots, E_m$. The form systems, then, are subalgebras of $M_n(\mathbf{R})$ or $M_n(\mathbf{C})$. (As we mentioned before, the relationship between hypercomplex systems and matrices, noted here by Frobenius, will play an important role in subsequent developments.) "Especially remarkable", Frobenius says, "are such systems of real forms for which the determinant of $\sum_{i=1}^{m} a_i E_i$ cannot vanish for real values of $a_1, a_2, \ldots, a_m$ without all these coefficients being identically zero". Frobenius thus singles out here for special attention the real division algebras. He then asks and answers the (more or less obvious) question: What are all of the real division algebras?

C.S. Peirce's proof of the above result on real division algebras appeared in one of the many notes he added to his father's paper "Linear Association Algebra" which he (C.S.) published in the *American Journal of Mathematics* in 1881 [67]. (B. Peirce originally published 100 copies of his work, in lithographed form, and sent them to his friends and mathematical acquaintances.) C.S. Peirce's statement of the theorem reads: "Ordinary real algebra, ordinary algebra with imaginaries, and real quaternions are the only associative algebras in which division by finites [i.e. by nonzero elements] always yields an unambiguous quotient." [1]) See [48], [54], [66] for details.

(c) Commutative algebras

The result we have in mind here is that a finite-dimensional associative and commutative algebra over **R** or **C**, without nilpotent elements, is a direct sum

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[1]) As we previously mentioned, the Cayley numbers (octaves) form an 8-dimensional real division algebra. It is, however, not associative, but is *alternative*: $(a^2)b = a(ab)$ and $a(b^2) = (ab)b$ for every a and b in the algebra. In 1950 E. Kleinfeld showed that (aside from the reals, complex numbers and quaternions) there are no other finite-dimensional alternative real division algebras (see [4], [54]). In 1958, Bott, Kervaire, and Milnor showed, using high-powered methods of differential topology, that the only finite-dimensional real division algebras over **R** (not necessarily alternative) have dimensions 1, 2, 4, or 8.

of a number of copies of either **R** or **C**. Thus one not only adds but also multiplies the elements of the algebra componentwise (when they are given as $\sum a_i e_i$). An immediate consequence of this result is that the only commutative division algebras over **R** are **R** or **C**. (This latter result also follows, of course, from that of Frobenius/Peirce in (b) above.)

The above characterization of commutative algebras over **R** and **C** was obtained, independently, by Weierstrass and Dedekind in the 1860s, although their works were published only in 1884-85 (see [80]). Both men were motivated, at least in part, by the following remark of Gauss, made in an 1832 paper on complex numbers [66]:

> The author [Gauss] has reserved for himself [the task] of working out more completely the subject, which in the present treatise is actually only occasionally touched upon. There then, too, the [following] question will find its answer: Why can the relations between things which present a multiplicity of more than two dimensions not furnish still other kinds of quantities permissible in the general arithmetic?

It is remarkable that Gauss seems to have anticipated here (as also, of course, in connection with fundamental developments in other branches of mathematics) the study of hypercomplex systems, and the fact that there are no systems analogous to **C** (i.e. fields) whose dimensions are greater than 2.

Dedekind's interest in commutative rather than general hypercomplex systems is understandable. In his fundamental work on ideal theory in algebraic number fields, Dedekind views the number field as an extension of the field of rational numbers, hence as a finite-dimensional algebra over the rationals. He exploits this point of view in his studies of algebraic number theory. To Dedekind, then, a finite-dimensional commutative algebra was a familiar object.

Dedekind's work helped to stimulate the deeper works of Molien and Cartan on the structure of more general types of algebras. This is part of the story to which we turn next.

IV. Structure of algebras

The first example of a noncommutative algebra was given by Hamilton in 1843. During the next forty years mathematicians introduced other examples of noncommutative algebras, began to bring some order into them and to single out certain types of algebras for special attention. Thus low-dimensional