II. Exploration

We might surmise that the geometric motivation was the desire to extend vectors in the plane to vectors in space (an extension which the quaternions, in a sense, accomplished); the algebra stemmed from a natural (from a mathematician's point of view) desire to extend number-pairs to triples, and, when this failed, to quadruples; the metaphysical connection with the ideas of Kant, which we mentioned above, was a factor in all of Hamilton's works in algebra; as for the poetry, we can do no better than to quote Weierstrass: "No mathematician can be a complete mathematician unless he is also something of a poet". See [35] for a detailed analysis.

For twenty two years following the invention of the quaternions, Hamilton was preoccupied almost exclusively with their application to geometry, physics, and elsewhere. To him the quaternions were the long-sought key which would unlock the mysteries of geometry and mathematical physics. The main importance of the quaternions, however, lay in another direction, namely in algebra (see [56]). Poincaré's tribute of 1902 is telling:

> Hamilton's quaternions give us an example of an operation which presents an almost perfect analogy with multiplication, which may be called multiplication, and yet it is not commutative... This presents a revolution in arithmetic which is entirely similar to the one which Lobachevsky effected in geometry.

We will explore some of the consequences of that revolution in the following section.

II. Exploration

Hamilton's quaternions at first received less than universal understanding and acclaim. Thus when Hamilton communicated his invention (discovery?) to his friend John Graves, the latter responded as follows [66]:

> There is still something in this system [of quaternions] which gravels me. I have not yet any clear view as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties.

Most mathematicians, however (including Graves) quickly came around to Hamilton's point of view. The floodgates were opened and the stage was set for the exploration of diverse "number systems", with properties which departed in various ways from those of the real and complex numbers.[1]) We give a brief outline of some of these.

(a) OCTAVES (Octonions, Cayley numbers) (1844)

Within three months of Hamilton's creation of the quaternions, John Graves outlined a system of "numbers" which he called "octaves". These are elements of the form $a_0 1 + a_1 e_1 + a_2 e_2 + \ldots + a_7 e_7 (a_i$ real numbers), with basis elements $1, e_1, e_2, \ldots, e_7$ satisfying

$$e_i^2 = -1, 1 \cdot e_i = e_i \cdot 1, e_i e_j$$
$$= -e_j e_i (i \neq j), e_i e_{i+1} = e_{i+3}, e_{i+3} e_i = e_{i+1}, e_{i+1} e_{i+3} = e_i .$$

(See [50], [54], [80] for details.) This 8-dimensional algebra contains the quaternions, and is thus also noncommutative. It is, moreover, also not associative (i.e. $(ab)c \neq a(bc)$). It does possess, however, along with the quaternions, the property of unique division; that is, every nonzero element of the algebra has an inverse. We call such algebras *division algebras*; they will play an important role in our story.

The octaves (or octonions) are nowadays known as Cayley numbers since Cayley, independently of Graves, published a note on them in 1845. (It was a postscript to a short paper on elliptic functions.) Neither Graves nor Cayley pointed out the nonassociativity of these "numbers".

(b) TRIPLE ALGEBRAS (1844)

De Morgan was quickly converted to the point of view that it is legitimate to create systems whose properties diverge from those of the real numbers. In a paper of 1844 entitled "On The Foundations of Algebra" he introduced several algebras of triples of real numbers. These were commutative but nonassociative, and De Morgan designated them "imperfect".

(c) EXTERIOR ALGEBRAS (1844)

In his *Ausdehnungslehre* of 1844 Grassmann attempted to construct algebraically an abstract science of "spaces", freed from spatial conceptualiza-

[1]) As noted by Poincaré (p. 233), there is an analogy between the creation of quaternions and the creation of non-Euclidean (hyperbolic) geometry more than a decade earlier. Both achievements were radical violations of prevailing conceptions. Moreover, both inspired constructions of various analogous systems (algebras and geometries, respectively) which eventually led to their systematic classification (cf. secs. III & IV below for algebra and Klein's "Erlangen Program" for geometry).

tion and restriction to three dimensions—that is, to construct a vector algebra of n-dimensional space. The fundamental notion in his theory is that of an "extensive quantity", that is an expression of the form $a_1 e_1 + \ldots + a_n e_n$, where $a_i \in \mathbf{R}$ (the real numbers) and e_i are linearly independent "units". On these quantities he defined various products, some of them yielding what are now known as exterior algebras. See [16], [19], [48], [50], [80] for details.

Grassmann's work was highly original but written obscurely and too abstractly to be appreciated at that time. A second, improved edition of his work, published in 1862, and a changing mathematical climate gained him the recognition he deserved.[1])

(d) BIQUATERNIONS (1853)

In his *Lectures on Quaternions* of 1853 Hamilton introduced "biquaternions", that is, "quaternions" with complex coefficients. He showed that they possess zero divisors (i.e. nonzero elements a and b such that $ab = 0$) and thus do not form a division algebra. In this work Hamilton also began consideration of "hypernumbers", namely n-tuples of real numbers.

(e) GROUP ALGEBRA (1854)

In 1854 Cayley published a paper entitled "On the theory of groups, as depending on the symbolic equation $\theta^n = 1$", in which he defined a (finite) abstract group. At the end of this paper Cayley gave the definition of a *group algebra* (of a finite group over the real or complex numbers). He called it a system of "complex quantities" and observed that it is analogous in many ways to Hamilton's quaternions (i.e. it is associative, noncommutative, but, in general, not a division algebra). See [49], [80].

(f) MATRICES (1855/1858)

In 1855, in a paper entitled "Remarques sur la notation des fonctions algébriques", Cayley introduced matrices, defined the inverse of a matrix and the product of two, and exhibited the relation of matrices to quadratic and bilinear forms. In an 1858 paper entitled "A memoir on the theory of matrices" he also defined the sum of matrices and the product of a matrix by a scalar, and showed (essentially) that $n \times n$ matrices form an associative algebra. In his own words [66]:

[1]) In this edition Grassmann explicitly mentions that his multiplication of extensive quantities applies, in particular, to yield the quaternions.

> It will be seen that matrices (attending only to those of the same order) comport themselves as *single quantities*; they may be added, multiplied or compounded together; the law of addition of matrices is precisely similar to that for the addition of ordinary algebraic quantities; as regards their multiplication (or composition), there is the peculiarity that matrices are not in general convertible [commutative].

In the same paper, Cayley shows that if L, M are 2×2 matrices such that $LM = -ML$, $L^2 = -1$, $M^2 = -1$, then letting $N = ML$ we get $N^2 = -1$, $MN = -NM$. He notes that L, M, N may serve as the i, j, k units of the quaternions, thus showing (essentially) that the quaternions are (isomorphic to) a subalgebra of the algebra of 2×2 matrices over **C** (the complex numbers).[1]) The relationship between hypercomplex number systems and matrices, of which this is an instance, was to be of fundamental importance for subsequent developments.

Cayley's introduction of matrices (as that of abstract groups around the same time) was well ahead of its time. (Frobenius, independently of Cayley, also discussed what amounts to the algebra of matrices, without using matrix notation, in a fundamental paper of 1878—see p. 240 below). Although Cayley appreciated the usefulness of matrices in simplifying systems of linear equations and composition of linear transformations, his broader concern for the abstract point of view in mathematics is apparent in much of his work. This was characteristic of a number of British mathematicians of the period, such as Peacock, De Morgan, Boole, Sylvester and, to some extent, Hamilton (who after his creation of the quaternions was at least partly converted to the symbolical-algebra point of view). See [41], [50], [66].

(g) CLIFFORD NUMBERS/ALGEBRA (1873/1878)

In a paper of 1873 entitled "Preliminary sketch of biquaternions," Clifford introduced, in connection with certain problems in geometry and physics, the so-called Clifford numbers (he called them "biquaternions", but they should be distinguished from Hamilton's biquaternions). These are elements of the form $q_1 + q_2\alpha$, where q_1, q_2 are quaternions, $\alpha^2 = 1$, and $\alpha q_i = q_i \alpha$. They form a nonassociative 8-dimensional algebra over **R** (not a division algebra).

In an 1878 paper entitled "Applications of Grassmann's extensive algebra", Clifford introduced what are known as "Clifford algebras"—associative $2n$-dimensional algebras, generated by n units $1, e_2, \ldots, e_n$, subject to the conditions $e_i^2 = -1$, $e_ie_j = -e_je_i$, and such that each product of two units is a new unit. See [48], [50], [80].

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[1]) In 1854 Cayley showed that every finite (abstract) group is isomorphic to a group of permutations. See [49].

(h) Nonions (1882)

Sylvester wrote many papers between 1882 and 1884 on hypercomplex number systems, matrices, and their connections. (In fact, Sylvester claimed that he had created the theory of matrices independently of Cayley.) One such example was his system of "nonions"—a 9-dimensional algebra over **R**, generated by elements $u^i v^j (i, j = 0, 1, 2)$, where

$$u = \begin{pmatrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0 \end{pmatrix},$$

ρ being a cubic root of 1. He showed that this algebra is isomorphic to the algebra $M_3(\mathbf{R})$ of 3×3 matrices over **R**.

III. Classification

What we have presented above is only a sample, albeit a representative one, of various hypercomplex number systems introduced, largely by British mathematicians, within 30 or 40 years of Hamilton's work on quaternions. The scene now shifted to the United States and to continental Europe. Now that a stock of examples of noncommutative number systems had been established, one could begin to have a theory. The general concept of a (finite-dimensional) associative algebra (a hypercomplex number system) emerged, and there was a move to classify certain types of these general structures. We focus on three such developments.

(a) Low-dimensional algebras

Of fundamental importance here is B. Peirce's groundbreaking paper "Linear Associative Algebra" of 1870. In the last 100 pages of this 150-page paper Peirce classifies algebras (i.e. hypercomplex number systems) of dimension < 6 by giving their multiplication tables. There are, he shows, over 150 such algebras! What is important in this paper, though, is not the classification but the means used to obtain it.[1] For here Peirce introduces concepts, and derives results, which proved fundamental for subsequent developments.

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[1]) Algebras of fixed (low) dimension were also classified, using different methods, by Scheffers, Study, and others. See [36], [38], [39], [66] for details. The complexity of the structure of general algebras, even of low dimensions, directed later researchers to focus on the study of special types of algebras (see e.g. (b) and (c) below and sec. IV).