§2. The extended dual state lemma

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The *writhe number* $w(K)$ of an oriented link diagram K is the sum of the signs over all crossing points of K. Little believed that the writhe number of an oriented reduced alternating diagram is a link type invariant. This conjecture has been recently proved independently by Murasugi [6] and Thistlethwaite [9]. It follows directly from the following Theorem.

THEOREM 2 (Murasugi [6]). *If K is an oriented weakly alternating diagram, then*

$$w(K) = \sigma(L) - d_{\max}(V_L(t)) - d_{\min}(V_L(t))$$

where the oriented link presented by K is denoted by L, its signature by $\sigma(L)$, and where $d_{\max}$ and $d_{\min}$ denote the maximal and minimal degrees of a polynomial. (Note that Murasugi uses the polynomial $V = V_L(t^{-1})$, so that his formula has two plus signs.)

Theorems 1 and 2 imply that, for oriented weakly alternating diagrams, both the number of positive crossing points and the number of negative crossing points are link type invariants.

It is worth realizing that, if $K^{\times}$ is the mirror image of an oriented link diagram K, then $w(K^{\times}) = -w(K)$. Therefore, if K is weakly alternating and represents an amphicheiral link, then Theorem 2 implies that $w(K) = 0$.

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The remaining part of this paper is organized as follows. In § 2 the extended dual state Lemma, due to Kauffman and Murasugi, is stated and proved. In § 3 I quickly recall the Kauffman state model for the Jones polynomial. Theorem 1 is proved in § 4 and Theorem 2 is proved in § 5. In the Appendix, the inequality (i) of Theorem 1 is somewhat improved.

§ 2. THE EXTENDED DUAL STATE LEMMA

Let Γ be the image of a generic immersion of a finite number of circles into R^2. Note that self-crossing points of Γ are exclusively double points. For each double point x of Γ a small disc in R^2 centered in x is divided by Γ into four parts. These parts appear in two pairs of opposite sectors. Each of these pairs is called a *marker* of Γ at x. In pictures these markers are indicated like that:

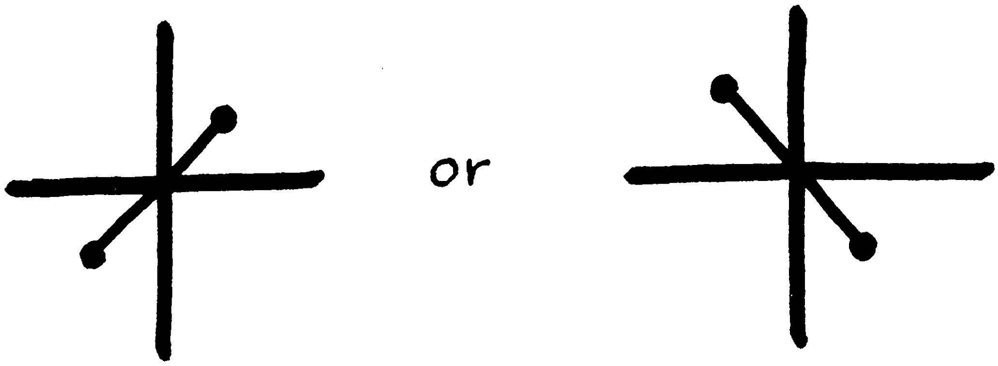


FIGURE 5

One can smooth (or surger) Γ along the markers:

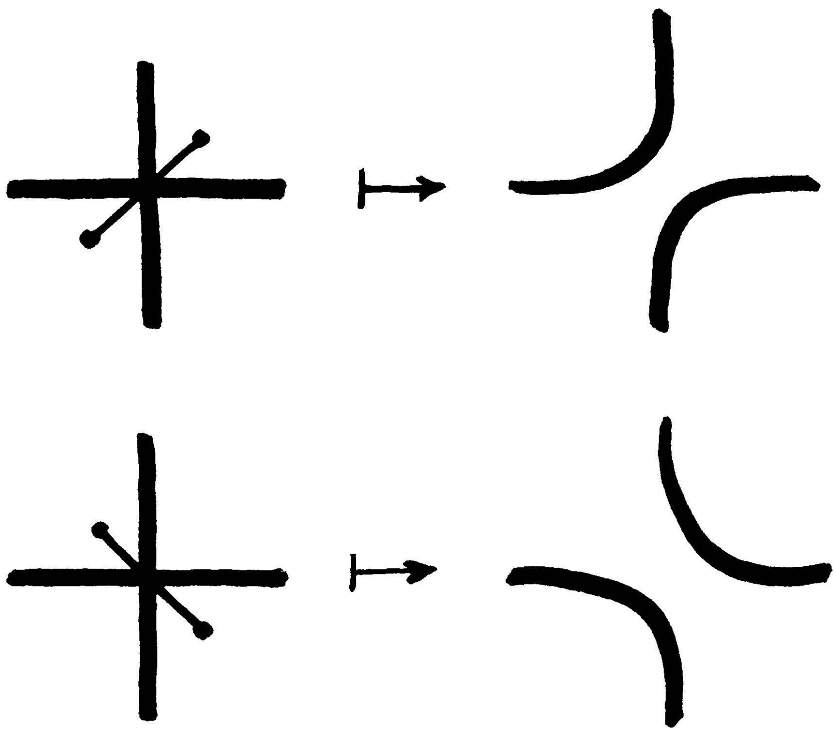


FIGURE 6

A *state* S for Γ is a choice of one marker at each double point of Γ. The opposite choice of marker at each double point defines the *dual state* of S, denoted by $\check{S}$. The dual state of $\check{S}$ is obviously S. If we surger Γ along the markers of a state S we obtain a closed imbedded 1-manifold $\Gamma_S \subset R^2$ as in the following picture.

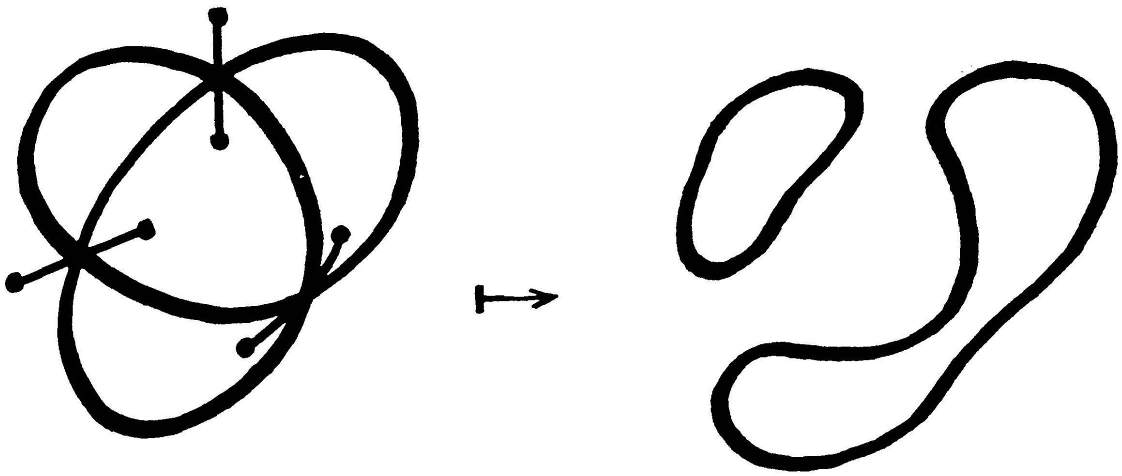


FIGURE 7

Let $|S|$ denote the number of connected components of Γ_S.

Denote by $r = r(\Gamma)$ the number of connected components of the set Γ in R^2, and by $c = c(\Gamma)$ the number of double points of Γ. It is clear that Γ has 2^c states. (If $c = 0$, then, by definition, Γ has one state S with $\Gamma_S = \Gamma$.)

LEMMA 1 (the dual state Lemma [5]). *For any state* S *of* Γ, *one has*

$$|S| + |\check{S}| \leqslant c + 2r. \tag{4}$$

To prove this Lemma and to study the case of equality in (4), we need the following definitions.

By an *edge* of Γ, we shall mean an arc in Γ whose interior does not contain any double point, and whose two ends are double points of Γ. The case of coinciding ends is not excluded, and such an edge is called a *loop*.

Let S be a state of Γ. An edge e of Γ is called *S-monochrome* if either e is a loop, or e has distinct ends and the markers of S at these ends look like this:

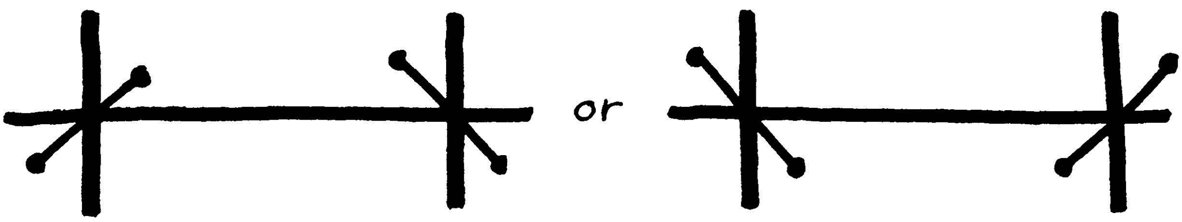


FIGURE 8

The edges of Γ which are not S-monochrome are called *S-polychrome.* Any *S-polychrome* edge has two distinct ends and the markers of S at these ends look like this:

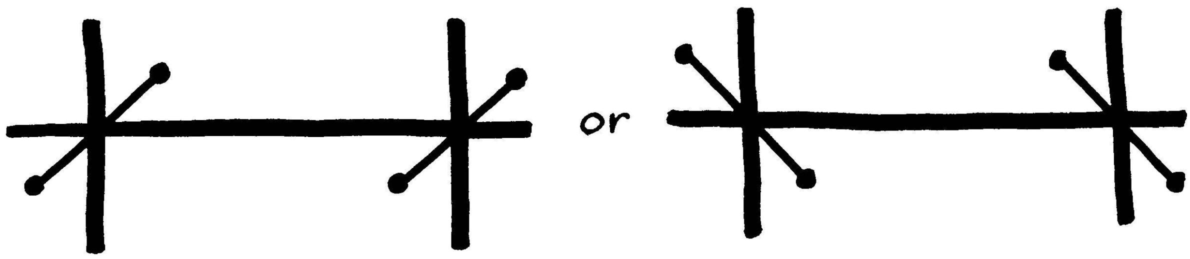


FIGURE 9

The state S of Γ is called *monochrome* if all edges of Γ are S-monochrome. It is clear that S is monochrome if and only if $\check{S}$ is monochrome.

We shall say that Γ is *prime* if each circle $S^1 \subset R^2$ which intersects Γ in exactly two points and transversally bounds a disc in $S^2 = R^2 \bigcup \{\infty\}$ which intersects Γ in a simple arc.

LEMMA 2. *Suppose that* Γ *is prime and connected. Let* S *be a state of* Γ. *Then the equality*

$$|S| + |\check{S}| = c + 2$$

holds if and only if S *is monochrome.*

Proof of lemmas 1 and 2. Let S be a state of Γ. To each double point x of Γ we associate a small square in R^2:

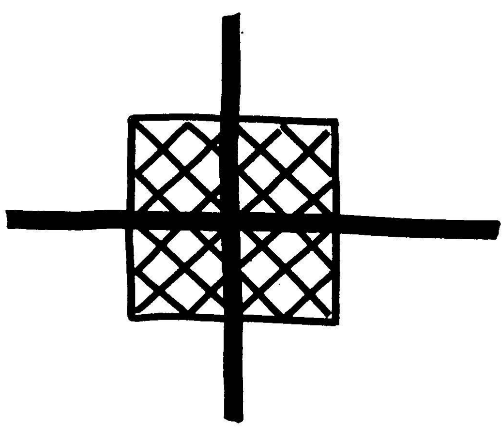


FIGURE 10

To each S-monochrome edge e of Γ we associate a plane band with core e:

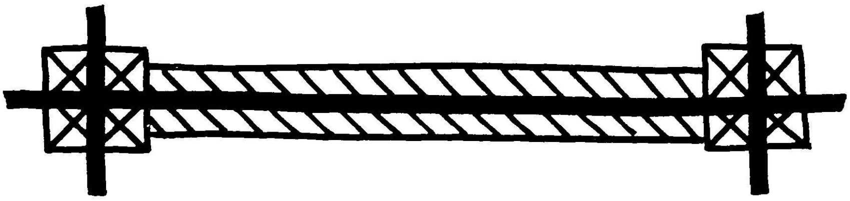


FIGURE 11

If e is a loop, the band looks like this:

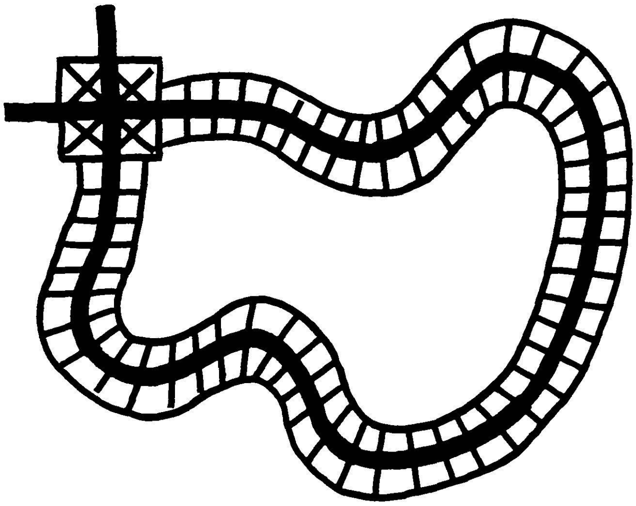


FIGURE 12

To each S-polychrome edge e we associate a 1-twisted band in R^3 with core e:

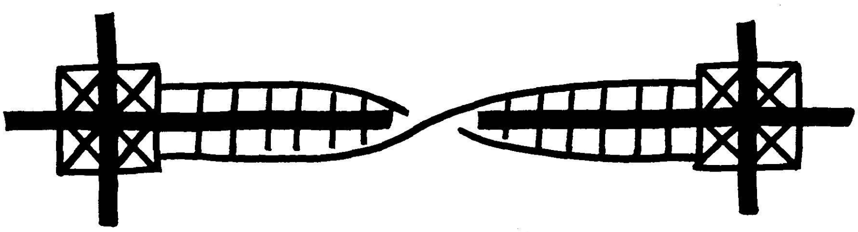
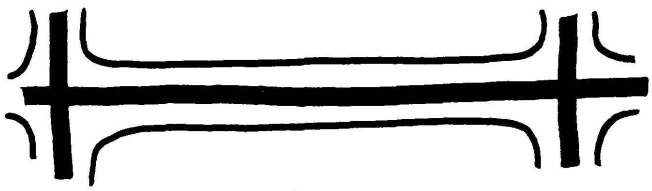


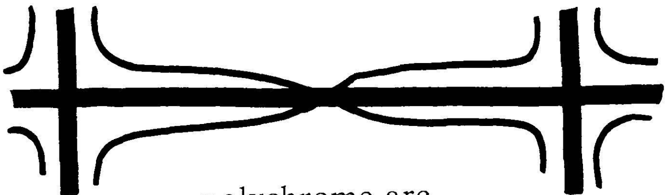
FIGURE 13

Denote by $M = M(S)$ the union of all these squares and bands. It is clear that M is a compact surface in R^3.

It is easy to check that the boundary ∂M of M is the disjoint union $\Gamma_S \amalg \Gamma_{\bar{S}}$, where it is understood that Γ_S and $\Gamma_{\bar{S}}$ are slightly moved away in R^3 to avoid intersections. See the following picture:



monochrome arc



polychrome arc

FIGURE 14

Therefore $|S| + |\check{S}| = b_0(\partial M)$ where b_i denote the i-th Betti number of a space with coefficients $\mathbf{Z}/2\mathbf{Z}$. As M retracts on Γ by deformation, $b_i(M) = b_i(\Gamma)$ for all i. In particular, $b_0(M) = r$. Since Γ is quadrivalent and has c double points, Γ has $2c$ edges. Thus

$$b_1(M) = b_0(M) - \chi(M) = r - (c-2c) = r + c\,.$$

Consider the homology exact sequence of the pair $(M, \partial M)$ with coefficients $\mathbf{Z}/2\mathbf{Z}$:

$$\ldots \to H_1(M) \to H_1(M, \partial M) \to H_0(\partial M) \to H_0(M) \to \{0\}\,.$$

As $b_1(M, \partial M) = b_1(M) = r + c$ by Poincaré duality, one has

$$|S| + |\check{S}| = b_0(\partial M) \leqslant b_0(M) + b_1(M, \partial M) = 2r + c\,.$$

This proves Lemma 1.

Let us now prove Lemma 2. The equality $|S| + |\check{S}| = c + 2$ holds if and only if the inclusion homomorphism $H_1(M) \to H_1(M, \partial M)$ in the exact sequence above is equal to zero. This happens if and only if the intersection form

$$H_1(M) \times H_1(M) \to \mathbf{Z}/2\mathbf{Z} \tag{5}$$

is zero. If S is monochrome then $M(S)$ is a planar surface, so that the form (5) is indeed zero.

Suppose that S is not monochrome. We shall prove that the form (5) is non zero. This will imply the strict inequality $|S| + |\check{S}| < c + 2$.

Let e be a S-polychrome edge of Γ. Consider the connected components of $R^2 - \Gamma$ which are adjacent to e. These components are distinct: Otherwise there would exist a simple loop in R^2 intersecting Γ in exactly one regular point, which is impossible. Denote these two components by a and b. It is clear that $\bar{a} \cap \bar{b}$ is a union of edges and double points of Γ, with in particular $e \in \bar{a} \cap \bar{b}$. If $\bar{a} \cap \bar{b}$ were to contain an edge of Γ distinct from e, then the dotted circle in the following picture would intersect Γ in two points.

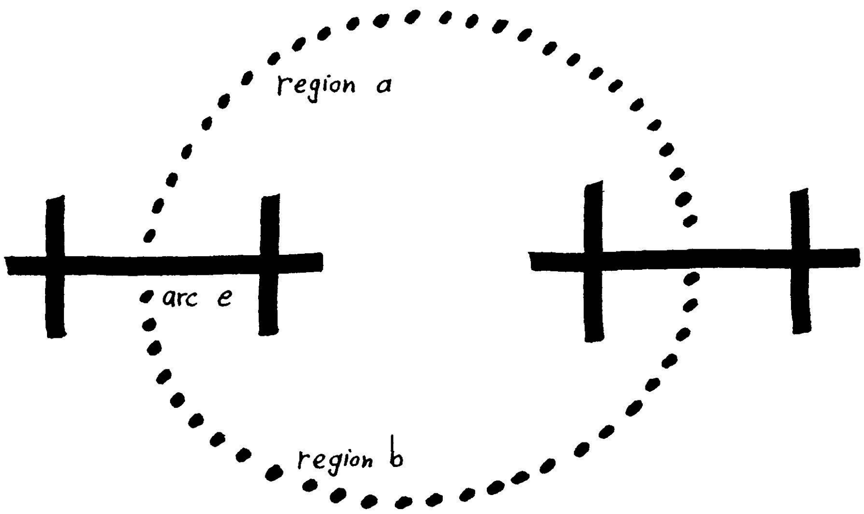


FIGURE 15

But this is impossible because Γ is prime. Thus $\bar{a} \cap \bar{b}$ is equal to the union of e and some double points.

Since e is S-polychrome, the intersection of the homology classes $[\partial \bar{a}]$ and $[\partial \bar{b}]$ in $H_1(M) \approx H_1(\Gamma)$ is equal to 1 (modulo 2):

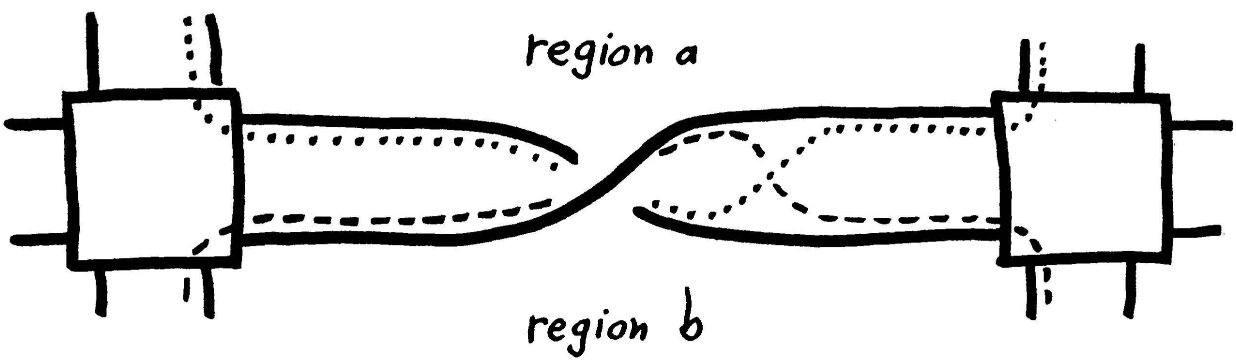


FIGURE 16

Thus (5) is a non-zero form, and the proof is complete. □

Remark. It is not important for us but curious to observe that $M(S)$ is always an orientable surface.