§1. Introduction

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A SIMPLE PROOF OF THE MURASUGI AND KAUFFMAN THEOREMS ON ALTERNATING LINKS

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The aim of the present paper is to give simplified proofs of several theorems recently obtained by Murasugi and Kauffman with the help of Jones polynomials for links. These theorems settle several old conjectures of Tait on alternating link diagrams. The proofs given here follow the main lines of the proofs given in [3], [6]; however some steps are considerably simplified, including the crucial "extended dual state Lemma".

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§ 1. INTRODUCTION

For the definition of (smooth) links in the 3-sphere, link diagrams, alternating diagrams and alternating links, the reader is referred to [3].

A link diagram is called *reduced* if there is no (smooth) circle $S^1 \subset R^2$ intersecting the diagram in exactly two points which lie near a crossing point, as in the following picture.

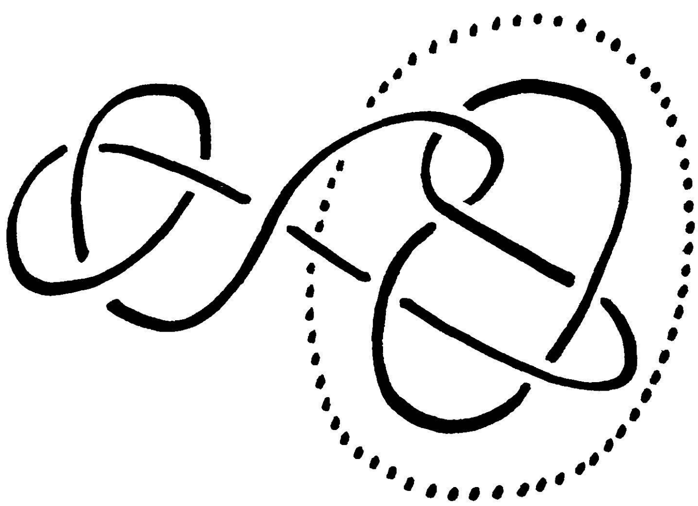

FIGURE 1

A link diagram is called *splittable* if there is a circle $S^1 \subset R^2$ which does not intersect the diagram and such that both components of $R^2 - S^1$ intersect the diagram. A link diagram K is said to be a *connected sum* of link diagrams $K_1, \ldots, K_m$ if $K_1, \ldots, K_m$ lie in disjoint discs in R^2 and if K can be obtained from $K_1, \ldots, K_m$ by band summation (the bands are supposed to lie in R^2 and to have no crossing point with each other and with $\bigcup_i K_i$). Finally, a link diagram is called *weakly alternating* if each of its split components is either a reduced alternating diagram or a connected sum of reduced alternating diagrams. Here is an example of a weakly alternating diagram which is not alternating.

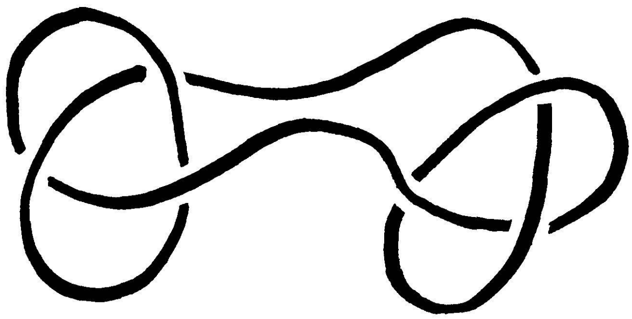

FIGURE 2

For a link diagram K we denote by $c(K)$ the *number of crossing points* of K and by $r(K)$ the *number of split components* of K.

Recall that with each oriented link $L \subset S^3$, V. Jones [4] has associated a polynomial $V_L(t) \in \mathbf{Z}[t^{1/2}, t^{-1/2}]$. If

$$V_L(t) = \sum_{n \leqslant i \leqslant m} a_i t^i \quad \text{with} \quad n, m, i \in \frac{1}{2}\mathbf{Z} \quad \text{and} \quad a_n \neq 0 \neq a_m ,$$

then one defines $\operatorname{span}(L) = m - n$.

According to [4], if L has an odd number of components, then $V_L(t) \in \mathbf{Z}[t, t^{-1}]$; if L has an even number of components, then $t^{1/2} V_L(t) \in \mathbf{Z}[t, t^{-1}]$. Therefore, in all cases $\operatorname{span}(L) \in \mathbf{Z}$. Note also that $\operatorname{span}(L)$ is not changed if we invert the orientations of some components of L (thanks to the Jones reversing result, see § 8 of [3]). Thus the integer $\operatorname{span}(L)$ is an invariant of non-oriented links.

This invariant has the following additive properties. If L splits into links $L_1, \ldots, L_r$ then

$$\operatorname{span}(L) = r - 1 + \sum_{i=1}^{r} \operatorname{span}(L_i) . \tag{1}$$

This follows from the formula

$$V_L(L) = (-t^{1/2}-t^{-1/2})^{r-1} \prod_{i=1}^{r} V_{L_i}(t)$$

of Jones [4]. If L is a connected sum of two links L' and L'' (performed from the unlinked union on any choice of components), then

$$V_L(L) = V_{L'}(t)V_{L''}(t)$$

so that

$$\text{span}(L) = \text{span}(L') + \text{span}(L'') .$$

THEOREM 1 (Murasugi, Kauffman). *Let K be a diagram of a link L. Then:*

(i) $c(K) + r(K) - 1 \geqslant \text{span}(L)$,

(ii) $c(K) + r(K) - 1 = \text{span}(L)$ *if and only if K is a weakly alternating diagram.*

In particular, as $r(K) = 1$ if L is unsplittable:

COROLLARY 1. *Let K be a diagram of an unsplittable link L. Then* $c(K) \geqslant \text{span}(L)$, *with equality if and only if K is a connected sum of reduced alternating diagrams.*

Let us observe that, if K and K' are alternating projections, one can always make connected sums K_1 and K_2 of K and K' in order that K_1 be alternating and K_2 be non-alternating. In particular, it follows that a link which has a weakly alternating projection is indeed an alternating link. See figure 3.

COROLLARY 2. *Two weakly alternating diagrams of the same alternating link L have the same number of split components. This number is equal to the number of split components of L.*

Proof. It is enough to note that every unsplittable weakly alternating diagram represents an unsplittable link. This fact is well known (at least for unsplittable alternating diagrams: see Crowell [1] and references therein). However, for the reader's convenience, we shall give here a proof of this fact which depends only on Theorem 1 and on a few elementary observations.

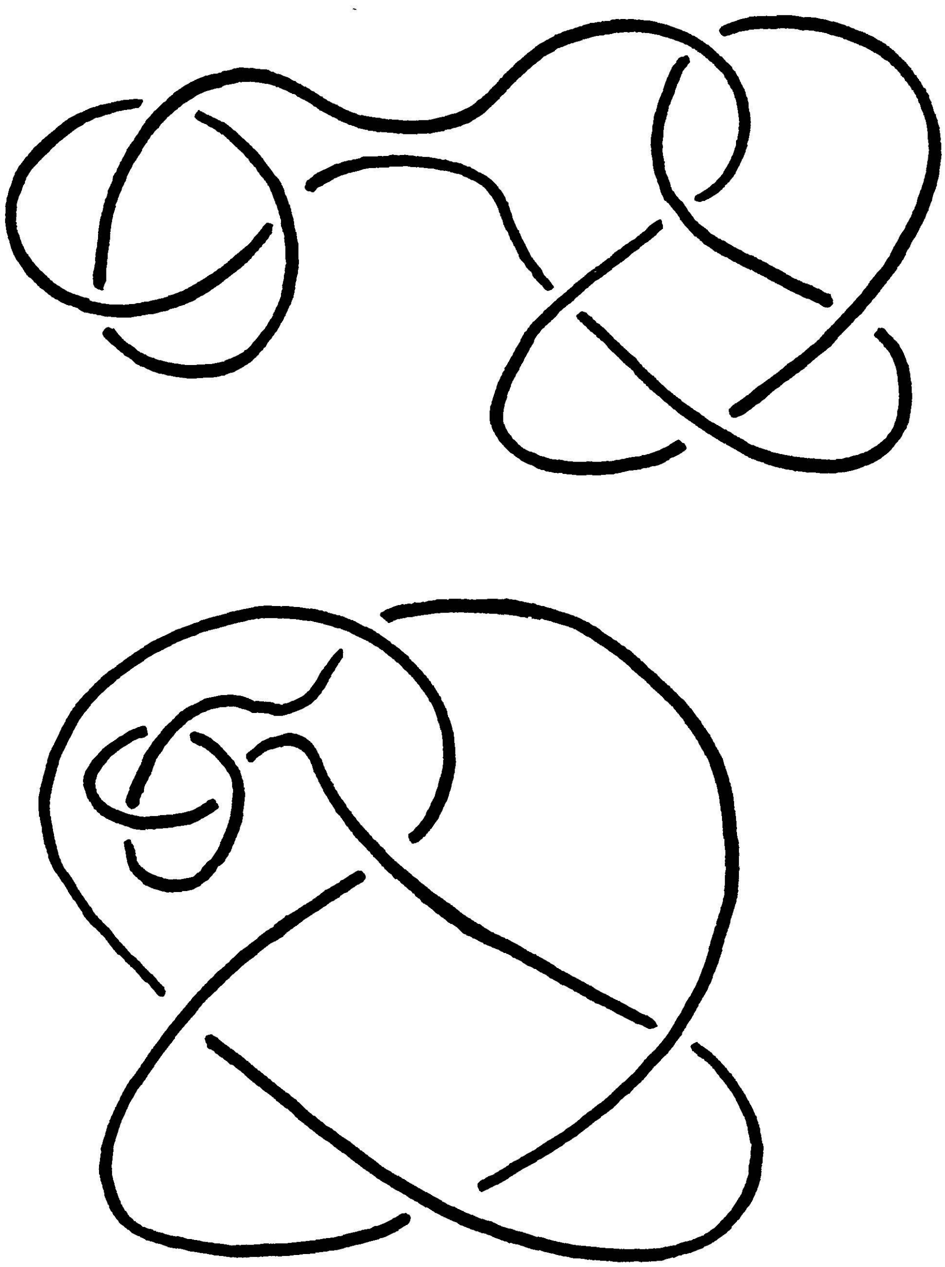

FIGURE 3

Let L be a link presented by an unsplittable weakly alternating diagram K, so that $r(K) = 1$. Suppose that L splits into unsplittable links $L_1, \ldots, L_p$. Then K is a "union" of subdiagrams $K_1, \ldots, K_p$ where K_i represents L_i for $i = 1, \ldots, p$. Since L_i is unsplittable, K_i is also unsplittable. In view of Corollary 1

$$(2) \qquad \sum_{i=1}^{p} c(K_i) \geqslant \sum_{i=1}^{p} \operatorname{span}(L_i) = \operatorname{span}(L) - (p-1) = c(K) - (p-1)\,.$$

Let us prove that

$$(3) \qquad c(K) \geqslant c(K_1) + c(K_2) + \ldots + c(K_p) + 2(p-1)\,.$$

Consider the graph with p vertices $k_1, ..., k_p$ in which the vertices k_i and k_j are connected by one edge if $i \neq j$ and if K_i crosses K_j in one point at least. Since K is unsplittable, this graph is connected. Thus it has at least $p - 1$ edges. On the other hand, the number of crossings of K_i and K_j is even for $i \neq j$. Therefore, if K_i crosses K_j at all, the number of such crossings is at least two. This implies (3).

Formulas (2) and (3) show that $p = 1$, namely that L is unsplittable. □

COROLLARY 3. *Two weakly alternating diagrams of the same alternating link L have the same number c of crossing points. This number is minimal among all diagrams of L. Any diagram of L with c crossing points is weakly alternating.*

Proof. This is straightforward from Theorem 1 and Corollary 2. Of course, $c = \text{span}(L) - 1 + r$ where r is the number of split components of L. □

Remarks.

1. In the case of alternating diagrams of knots, the first two statements of Corollary 3 were conjectured by Tait [8]. For a recent discussion of this and of other conjectures by Tait, see [3].

2. For non-alternating link diagrams, the inequality (i) of Theorem 1 can be somewhat improved — see the Appendix to the present paper.

The next theorem is concerned with the writhe number of an oriented alternating link diagram. Recall that, up to isotopy in R^2, there are two types of crossing point of oriented link diagrams, distinguished by a sign:

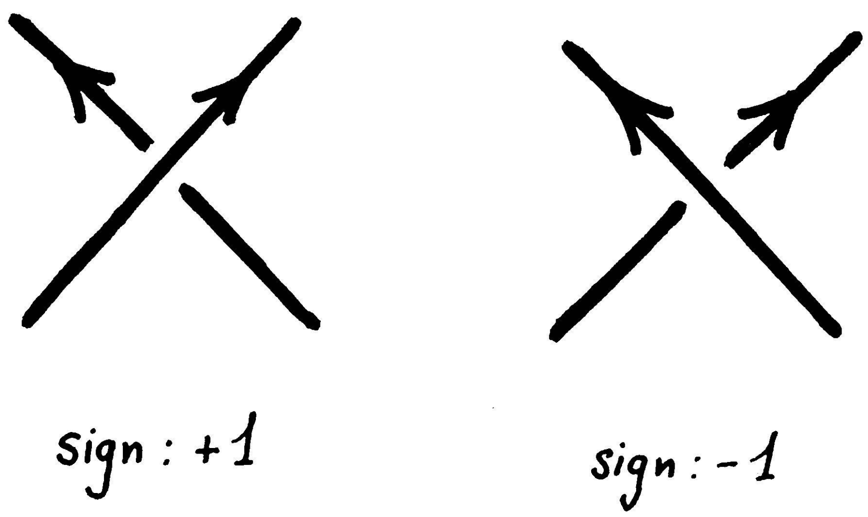

FIGURE 4

The *writhe number* $w(K)$ of an oriented link diagram K is the sum of the signs over all crossing points of K. Little believed that the writhe number of an oriented reduced alternating diagram is a link type invariant. This conjecture has been recently proved independently by Murasugi [6] and Thistlethwaite [9]. It follows directly from the following Theorem.

THEOREM 2 (Murasugi [6]). *If K is an oriented weakly alternating diagram, then*

$$w(K) = \sigma(L) - d_{\max}(V_L(t)) - d_{\min}(V_L(t))$$

where the oriented link presented by K is denoted by L, its signature by $\sigma(L)$, and where $d_{\max}$ and $d_{\min}$ denote the maximal and minimal degrees of a polynomial. (Note that Murasugi uses the polynomial $V = V_L(t^{-1})$, so that his formula has two plus signs.)

Theorems 1 and 2 imply that, for oriented weakly alternating diagrams, both the number of positive crossing points and the number of negative crossing points are link type invariants.

It is worth realizing that, if $K^{\times}$ is the mirror image of an oriented link diagram K, then $w(K^{\times}) = -w(K)$. Therefore, if K is weakly alternating and represents an amphicheiral link, then Theorem 2 implies that $w(K) = 0$.

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The remaining part of this paper is organized as follows. In § 2 the extended dual state Lemma, due to Kauffman and Murasugi, is stated and proved. In § 3 I quickly recall the Kauffman state model for the Jones polynomial. Theorem 1 is proved in § 4 and Theorem 2 is proved in § 5. In the Appendix, the inequality (i) of Theorem 1 is somewhat improved.

§ 2. THE EXTENDED DUAL STATE LEMMA

Let Γ be the image of a generic immersion of a finite number of circles into R^2. Note that self-crossing points of Γ are exclusively double points. For each double point x of Γ a small disc in R^2 centered in x is divided by Γ into four parts. These parts appear in two pairs of opposite sectors. Each of these pairs is called a *marker* of Γ at x. In pictures these markers are indicated like that: