4. Chern Number Inequalities

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$= \mathbf{P}(H^d + (1))$ have Kähler metrics with positive holomorphic sectional curvature. On the other hand, for $d \geqslant 3$, M_d does not have positive first Chern class.

4. Chern Number Inequalities

In 1976, the author proved the Calabi conjecture and demonstrated the following Chern number inequality for algebraic manifolds with either ample or trivial canonical line bundles:

$$(*) \qquad (-1)^n c_2 c_1^{n-2} \geqslant \frac{(-1)^n}{2(n+1)} c_1^n$$

where equality holds if and only if M is covered by the ball, i.e., $M = B/\Gamma$ for some $\Gamma \subseteq SU(n, 1)$. Around the same time, Miyaoka [M3], extending the method of Bogomolov, obtained the same inequality for $n = 2$ under the weaker assumption that the Kodaira dimension of the surface is non-negative. However, he has not shown that equality holds if and only if M is covered by the ball.

By studying surfaces with singularities, Cheng and Yau [C-Y2] proved inequality (*) for surfaces of general type (equality holds if and only if M^2 is covered by the ball). The arguments in [C-Y2] can also be generalized to higher dimensions. One can also characterize surfaces M which are biholomorphic to B^n/Γ where $\Gamma \subseteq SU(2, 1)$ is allowed to have fixed points. Note that M is, in general, a variety since Γ may have fixed points.

It is also interesting to study manifolds which satisfy certain Chern number inequalities. Surfaces which satisfy inequality (*) have been studied by Hirzebruch, Deligne, Mostow, etc. A corollary of [Y2] is the following rigidity theorem for Kählerian structures on $\mathbf{CP}^n$: The only Kählerian structure on $\mathbf{CP}^n$ is the standard one; moreover, the only complex structure on $\mathbf{CP}^2$ is the standard one. For n odd, this result was due to Hirzebruch and Kodaira [H-K].

We now sketch the proof of inequality (*) when the canonical line bundle K of M is ample. In this case, there exists a Kähler-Einstein metric on K. For Kähler-Einstein metrics one observes that the Chern integral associated to the left hand side of (*) can be expressed in terms of the length squared of the curvature tensor. Since the Ricci tensor is the only part of the curvature tensor, the right hand side, which can be written as the determinant of the Ricci tensor, can be dominated by the left hand side.

If equality holds for (*), one sees that the integrands of both sides are equal. This last fact turns out to be equivalent to M having constraint holomorphic sectional curvature. Hence equality holds in (*) if and only if M is covered by the ball.

Kähler-Einstein metrics do not exist on algebraic manifolds whose canonical line bundle is not a multiple of some ample line bundle. However, it is still possible to study the inequality (*) for algebraic manifolds whose canonical line bundle is almost ample. In [Y1] it was proven that there exists a Kähler-Einstein metric which is degenerate along the divisor where the canonical line bundle is trivial. Similarly one can require the metric to blow up in a certain way. This fact was used by Cheng and Yau [C-Y2] to prove the inequality (*) for surfaces of general type.

$$(**) \quad c_1(M) \leqslant 0 \text{ on } M, \quad \text{and} \quad c_1(M) < 0 \text{ outside a subvariety of } M\,.$$

Recall that the Kodaira dimension $K(M)$ is defined by

$$K(M) = \begin{cases} -\infty & \text{if} \quad N(M) = 0 \\ \max\dim\{\phi_{mk}\}(M) & \text{if} \quad N(M) \neq 0 \end{cases},$$

where $N(M) = \{m > 0 \mid H^0(M, K^m) = 0\}$ and ϕ_{mk} is the pluricanonical mapping. It is easy to see that $K(M) \leqslant$ the algebraic dimension of $M \leqslant n$. If $K(M) = n$, then M is called a manifold of general type.

In dimension two, surfaces can be classified bimeromorphically by their Kodaira dimension. The surfaces with $K(M) = -\infty$, 0 or 1 are well understood; moreover, $K(M) = 2$ (i.e., M is a surface of general type) if and only if M satisfies (**). Suppose M is a three-fold of general type and K is the canonical line bundle divisor. Kawatama [Ka] proved that if $K \cdot C \leqslant 0$ for every algebraic curve $C \subseteq M$, then M satisfies (**).

Most likely (**) always implies (*); that is, if M^n is an algebraic manifold with almost ample canonical line bundle, then the inequality (*) holds. This is not known for $n \geqslant 3$. One would also like to know what the relationship is between manifolds of general type and the inequality (**). In this respect, consider the following theorem of Siu [S5]. First recall that Siegel's theorem [Sg] says that for a complex manifold M^n, the transcendence degree of the meromorphic function field of M over $\mathbf{C}$ is less than or equal to n. When equality holds, M is called a Moishezon manifold. A Moishezon manifold can always be obtained by blowing up and down an algebraic manifold a finite number of times and hence is birational to some projective algebraic manifold. For a Moishezon manifold, there always exists a holomorphic vector bundle L over M such that

$c_1(L) \geqslant 0$ on M and $c_1(L) > 0$ outside some subvariety of M. Siu [S5] proved that the converse is also true under the weaker assumption that $c_1(L)$ is nonnegative everywhere and positive at some point. Thus, a manifold which satisfies (**) is Moishezon. It is also not known whether $\mathbf{CP}^n$, $n \geqslant 4$, can admit a nonstandard structure which is Moishezon. For $n = 3$, T. Peternell [Pe] proved that if M is a Moishezon 3-fold which is topologically isomorphic to $\mathbf{CP}^3$, then M is the standard $\mathbf{CP}^3$. His proof depends heavily on Mori's theory of extremal rays in 3-folds. One might expect that it is helpful for this problem to study rational curves in a Moishezon manifold which is a topological $\mathbf{CP}^n$.

5. Kähler-Einstein Metrics on Noncompact Manifolds

We now consider Kähler-Einstein metrics on complete noncompact manifolds. Let g be a complete Kähler-Einstein metric on M^n, i.e., $R_{i\bar{j}} = cg_{i\bar{j}}$ for some constant c. If $c > 0$, Myer's theorem would imply N is compact. Hence, $c \leqslant 0$ and $c_1(M) \leqslant 0$. In this section we consider the case $c_1(M) < 0$ and leave the case $c_1(M) = 0$ for the next section.

One would like to characterize noncompact manifolds which admit complete Kähler-Einstein metrics $g_{i\bar{j}}$ with $R_{i\bar{j}} = -g_{i\bar{j}}$. In particular, one would like to impose conditions on M to guarantee the existence and uniqueness of a Kähler-Einstein metric. First of all, uniqueness always holds. That is to say, if M and N are complete Kähler-Einstein manifolds with $R = -1$ and $F: M \to N$ is a biholomorphism, then F is an isometry. To prove this, let g and dv and g' and dv' denote the Kähler-Einstein metrics and volume forms of M and N, respectively. If we let $\rho = \log(F^*dv'/dv)$, then $\partial\bar{\partial}\rho = -f^*\operatorname{Ric}' + \operatorname{Ric} = F^*g' + g$. Taking traces, we have $\Delta\rho = -n + n \cdot e^{\rho/n}$. Hence, the maximum principle implies $\rho \leqslant 0$ and $F^*dv' \leqslant dv$. Replacing F by F^{-1}, we have $F^*dv' \geqslant dv$ and F is an isometry.

Uniqueness also holds for "almost" complete Kähler-Einstein metrics with scalar curvature equal to minus one. Here, a metric ds^2 on M is said to be almost complete if we can write M as an increasing union of domains Ω_α and there exist complete metrics ds_α^2 on Ω_α for each α such that ds_α^2 converges to ds^2 on compact subsets of M. See Cheng-Yau [C-Y1] for details.

We now consider the existence of Kähler-Einstein metrics with negative scalar curvature. Of course, the existence of such a metric would give restrictions on the complex structure of M. For example, Eiseman [Ei] proved that if there exists a Hermitian metric with scalar curvature less than