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of f is not less than 4 at some point. The assumption of N being strongly negatively curved is similar to the negativity of the curvature operator. One expects to be able to weaken this condition. But, if one only assumes negative bisectional curvature, the analog of Siu's theorem is false. This is because for $M = B^n/\Gamma$ embedded in $\mathbf{CP}^n$ as a regular subvariety, any hyperplane section of M has negative bisectional curvature and it is not rigid in general.

Recently, Jost-Yau [JY1, 2] looked at the complex structure of complex surfaces M homotopy equivalent to $N = D \times D/\Gamma$ where Γ is irreducible. Let $f: M \to N$ be a harmonic homotopy equivalence where M is Kähler. By analyzing the foliation $f^\alpha \equiv$ const., they showed that the universal cover of M is biholomorphic to $D \times D$.

Subsequently, Mok [Mk2] generalized the theorem of Jost-Yau to arbitrary dimension. He also considered the foliation studied by Jost and Yau.

A generalization of the rigidity theorem to quasi-projective manifolds was made by Jost-Yau. They study the complex structure over Hermitian symmetric spaces with finite volume.

For a compact manifold M with strongly nonpositive curvature, one likes to prove M is either locally Hermitian symmetric or that the complex structure is rigid. Sampson [Sa] treated the case where M is Kähler and N is a Riemannian manifold with Hermitian negative curvature, that is $R^N_{ijkl} u^i v^j \bar{u}^k \bar{v}^l \leqslant 0$. By applying Bochner's technique in essentially the same way as Siu, he showed that all harmonic maps between M and N are holomorphic. Using Sampson's result, combined with the existence theorem for harmonic maps, we can easily obtain restrictions on the fundamental group of a Kähler manifold.

Another interesting situation is when M and N are Kähler manifolds and N has positive sectional curvature. Is it true that any minimizing harmonic map is holomorphic or antiholomorphic? This is only known when $M = \mathbf{CP}^1$. Also, if we can prove this assuming in addition that N is an irreducible symmetric space, then the conjecture that an irreducible symmetric Kähler manifold has only one Kähler structure is probably true. Notice that for the reducible Kähler manifold $\mathbf{CP}^1 \times \mathbf{CP}^1$, there exists infinitely many complex structures which are Kähler.

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The classification theory of harmonic maps from surfaces to Riemannian manifolds, especially symmetric spaces, is of interest to mathematical physicists. The simplest symmetric spaces are the real and complex projective spaces. In [Ca1], Calabi gave an effective parametrization of isotropic harmonic maps from surfaces into real projective space. Following Calabi and the work of physicists, Eells and Wood [EW2] set up a bijective correspondence between full isotropic harmonic maps $\phi: M^2 \to \mathbf{CP}^n$ and pairs (f, r) where $f: M^2 \to \mathbf{CP}^n$ is a full holomorphic map and $0 \leqslant r \leqslant n$ is an integer (see [Ca1] and [EW2] for definitions). Their idea is based on the fact that if $\phi: M \to \mathbf{CP}^n$ is a full isotropic map, then for some $r, s, r + s = n$, the map

$$f = [(\phi \oplus D'' \phi \oplus \cdots \oplus (D'')^{r-1} \phi \oplus (D' \phi \oplus \cdots \oplus (D')^s \phi)]^{\perp}$$

is full holomorphic. Here D' and D'' are the $(1, 0)$ and $(0, 1)$ components of the covariant derivative.

Later, Bryant ([Br1], [Br2]) treated conformal harmonic maps from surfaces into S^6 and S^4. Inspired by the twistor construction of Calabi and Penrose, he considered a restricted class of conformal harmonic maps, namely superminimal surfaces. (Note that Hopf already studied these surfaces in its primitive form). He established a one-to-one correspondence between superminimal surfaces and curves horizontal in $\mathbf{CP}^3$ with respect to the twistor fibration $\mathbf{CP}^3 \xrightarrow{T} S^4$. By constructing such a curve, Bryant showed that any Riemann surface be conformally immersed as a minimal surface in S^4. For the construction in a general 4-manifold, see [ESa].

Recently, K. Uhlenbeck [U3] has dealt with the space H of harmonic maps from a simply-connected 2-dimensional domain into a real Lie group $G_{\mathbf{R}}$ (which is the chiral model in the language of theoretical physics). She studied the algebraic structure of the manifold H and its relation with Kac-Moody algebras.

Another uncultivated area in harmonic maps is the classification of harmonic maps from a surface into a Ricci flat Kähler three-fold. The interest in this comes from the study of superstring theory in theoretical physics.

§ 4. Minimal submanifolds

The study of minimal submanifolds is another important topic in differential geometry. In this section we will mainly consider minimal surfaces in compact three manifolds. The minimal surfaces will be assumed to be regular and embedded, except when otherwise indicated.