2. Noncompact manifolds

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **33 (1987)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **19.09.2024**

is a manifold with boundary. This type of argument breaks down when we drop the non-positivity condition. For example Eells and Wood [EW1] have shown that there does not exist a degree 1 map from a 2-torus to a 2-sphere.

Instead of looking for harmonic maps in a homotopy class, one can look for harmonic maps with the same action on π_1. We say that two maps $f, g: M \to N$ are π_1- equivalent if $f_* = g_*: \pi_1(M) \to \pi_1(N)$. When M is a Riemann surface, L. Lemaire [Lm] proved the existence of a regular, energy minimizing harmonic map in the class of π_1-equivalent maps.

Another treatment of this problem was given by Sacks-Uhlenbeck [SaU] and R. Schoen-S. T. Yau [Sc-Y1]. Schoen-Yau considered the function space L_1^2 and showed that for $u \in L_1^2(M, N)$, u_* is well-defined and preserved under the weak limit. Using the class $\{f \in L_1^2(M, N) \mid f_* = (f_0)_*\}$ which is weakly closed, combined with the regularity of minimizing harmonic maps from a surface, one can show the existence of a smooth harmonic map in this class.

Schoen-Yau's argument could be generalized to higher dimensions by restricting the map f to the two skeleton of M. (This was also observed by White [Wh].) It is reasonable to expect that one can produce an energy minimizing harmonic map whose action on $\pi_2(M)$ has some resemblance to a given map.

For minimizing harmonic maps, R. Schoen and K. Uhlenbeck [ScU1, 2] have done fundamental work. By delicate use of comparison maps, they showed that the Hausdorff dimension of the singular set of energy minimizing harmonic maps is of codimension at least three. Their theorem can be used to recover the former theorems of Eells-Sampson and Sachs-Uhlenbeck.

2. Noncompact manifolds

The theory for harmonic maps between noncompact manifolds is more complicated than when the manifolds are compact. One reason is that when we choose a minimizing sequence of maps, their energies may not be concentrated in a bounded region. On the other hand, one hopes that this can be prevented by making suitable topological assumptions on the manifolds.

For L^2-harmonic maps, i.e., weakly harmonic maps with finite energy, one can sometimes prove existence by making geometric or topological restrictions. When N is a manifold with nonpositive curvature, Schoen and

Yau [Sc-Y2] have generalized Eells-Sampson's [ES] and Hartman's [Hr] work. They showed that if N is a compact manifold with nonpositive sectional curvature, M is complete and $f: M \to N$ has finite energy, then f is homotopic on compact sets to a harmonic map with finite energy.

Later [Sc-Y3], by explicitly computing the hessian of the distance function d^2 considered as a function on $N \times N$, showed that the set of harmonic maps in a homotopy class is connected (see [Hr] when M is compact) and can be immersed in N as a totally geodesic submanifold. Moreover, it is a point if $\pi_1(N)$ has no nontrivial abelian subgroup and the image of M is neither a point nor a circle. Here we assumed M has finite volume and the harmonic maps have finite energy. (When N is locally symmetric, this is also done by Sunada.) They also applied the theory of harmonic maps to study finite groups acting on a compact manifold.

3. Rigidity

It is natural to ask if harmonic homotopy equivalences are isometries when M and N are both negatively curved Einstein manifolds with dimension $\geqslant 3$. This is based on the uniqueness of harmonic maps into negatively curved manifolds and the Mostow rigidity theorem. If this is true, it would give another proof of the Mostow rigidity theorem in the case of rank one symmetric spaces.

It is a question for negatively curved manifolds M and N, whether a harmonic homotopy equivalence is a diffeomorphism or not. Schoen-Yau [Sc-Y4] and Sampson [Sa] have proved this when M and N are Riemann surfaces. If we only assume non-positivity of curvature, Calabi has constructed a counterexample when N is a torus.

By minimizing the energy among diffeomorphisms, combined with a replacement argument, Jost-Schoen [JS] constructed a harmonic diffeomorphism between surfaces of the same genus without any curvature assumption. (Hence it generalizes a theorem of Schoen-Yau where one assumes the image has non-positive curvature.)

There are plenty of examples of harmonic maps when M and N are Kähler manifolds. In particular, holomorphic maps are harmonic. On the other hand, it was conjectured by Yau that when N has negative curvature, harmonic maps are holomorphic. In attempting to settle this conjecture of Yau, Siu [S2], proved that a harmonic map f is either holomorphic or antiholomorphic provided N is strongly negatively curved and the rank