§2. Yamabe's Equation and Conformally Flat Manifolds

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In the other direction, it is a major problem to construct bounded holomorphic functions on a complete simplete connected Kähler manifold with strongly negative curvature. In fact, one would like to prove that it is biholomorphic to a bounded domain in $\mathbf{C}^n$ or at least that bounded holomorphic functions separate points of the manifold. It looks like the problem is very much related to a possible generalization of the classical Corona problem to higher dimensional bounded domains.

§ 2. Yamabe's Equation and Conformally Flat Manifolds

Yamabe's equation is a nonlinear elliptic scalar equation related to the conformal deformation of a metric on a Riemannian manifold. Given a metric g_0 with scalar curvature R_0, let g be a metric pointwise conformal to g_0. Then $g = u^{4/(n-2)}g_0$, where $u > 0$ is a smooth function. The scalar curvature R of g is given by the equation

$$L_0 u = -\gamma_0 \Delta_0 u + R_0 u = R u^{\alpha}, \tag{1}$$

where Δ_0 is the Laplacian with respect to g_0, $\gamma_0 = \dfrac{4(n-1)}{n-2}$, $\alpha = \dfrac{n+2}{n-2}$ and $n = \dim M$.

In [Ya], Yamabe asserted that there is always a solution $u > 0$ to equation (1) with $R =$ const. That is to say, any metric on a compact Riemannian manifold is conformally equivalent to a metric with constant scalar curvature. However, his proof contained an error. This was discovered by Trudinger. Moreover, Trudinger [Tr] showed that (1) could be solved for $R =$ const. provided the lowest eigenvalue λ_1 of the linear operator L_0 is nonpositive.

Let Y be the functional on $L_1^2(M)$ defined by

$$Y = \int_M (\gamma \,|\, \nabla_0 u \,|^2 + R_0 u^2) \Big/ \left(\int_M R u^{\alpha+1} \right)^{\frac{2}{\alpha+1}}$$

where ∇_0 is the gradient with respect to the metric g_0. By a simple computation, one finds that (1) is the Euler-Lagrange equation for the functional Y.

Aubin [Au1] gave a sufficient condition for Y to have a minimum in $L_1^2(M)$. It can be described as follows. Fix $R \equiv 1$ and let $\sigma(g_0)$ be the minimum of Y, $\Lambda_n = \sigma(\hat{g})$ where $\hat{g}$ is the standard metric on the unit sphere S^n. Then

(a) $\sigma(g_0) \leqslant \Lambda_n$ for any metric g_0,

(b) If $\sigma(g_0) < \Lambda_n$, there exists a smooth function u minimizing Y.

Since u is a solution to (1) with $R = \text{const.}$, Yamabe's conjecture translates to whether or not $\sigma(g_0) < \Lambda_n$ for metrics not conformal to the standard metric on S^n. Aubin [Au1] proved that if $n \geqslant 6$ and g_0 is not conformally flat, then $\sigma(g_0) < \Lambda_n$. The argument of Aubin is local. He constructed a function supported in a small open set which is radial. Thus, the remaining cases are when $n = 3, 4$ or 5 and when M is locally conformally flat for $n \geqslant 6$.

Recently, R. Schoen [Sc] gave a complete solution to Yamabe's conjecture. His argument is global and uses the generalized positive mass theorem ([ScY5]), Schoen gave a higher order estimate for $Y(u^\varepsilon)$ for a suitable sequence $\{u^\varepsilon\}$ in the case where M is conformally flat or $n = 3$. The case $n \geqslant 4$ requires perturbation arguments using again the positive mass theorem.

We may also consider the same questions for complete, noncompact manifolds. Recently, Schoen announced some new results. A particularly interesting result is as follows. If M has the topological type of $S^n - \{p_1, \ldots, p_k\}$ for $k > 1$, then one can find a metric with scalar curvature equal to one in each conformal class of complete metrics.

Another topic related to the Yamabe conjecture is the study of (locally) conformally flat manifolds. A theorem of Kuiper [Ku] says that for any conformally flat, simply connected manifold M, one can find an open conformal mapping from M into the standard sphere which is unique up to a conformal diffeomorphism of S^n. This map is called the developing map. We denote its image by Ω and let $\Lambda = S^n - \Omega$.

Schoen and Yau [ScY6] obtained results relating the Hausdorff dimension of Λ to the sign of the scalar curvature of M. The results can be stated as follows.

1. If M is a complete (possibly compact) conformally flat manifold with positive scalar curvature $R \geqslant 1$, then the developing map is a conformal diffeomorphism into S^n. Hence, conformally, M is covered by an open subset of S^n. The argument here uses crucially the Green's function of the conformal operator.

2. If M is a compact conformally flat manifold with positive scalar curvature, then $\mu_{\frac{n}{2}-1}(\Lambda) = 0$. Here μ_k is the k dimensional Hausdorff measure.

3. If M is a compact Riemannian manifold covered conformally by $\Omega \subset S^n$ with $\mu_{\frac{n}{2}-1}(\Lambda) < \infty$, then M admits a metric with scalar curvature $R \geqslant 0$ in the same conformal class. It is conjectured that if $R \geqslant 0$, then $\mu_{\frac{n}{2}-1}(\Lambda) < \infty$.

The basic idea is that by using the developing map, we can reduce the problems to the study of a scalar equation, namely the Yamabe equation on an open subset of S^n. The remaining parts of the proofs are relatively easy. By using the same technique, Schoen and Yau proved that for a compact conformally flat manifold with positive scalar curvature, $\pi_i(M) = 0$ for $2 \leqslant i \leqslant n/2$. Some of their results are also valid for complete manifolds.

§ 3. Harmonic Maps

Harmonic maps are important objects in geometry and analysis. They appear naturally as critical points of an energy functional of the appropriate function space. Harmonic maps reflect a lot about the geometric properties of manifolds.

Given Riemannian manifolds M and N, consider the mapping space $C^r(M, N)$. One problem is to find nice (i.e., canonical) representatives in this space. For a map $f: M \to N$ we define its energy by $E(f) = \int_M |df|^2 dV_M$.

A harmonic map is a critical point of this energy. The first question is that of existence, uniqueness and regularity.

1. Existence, uniqueness and regularity

The first major work was done by J. Eells and L. Sampson [ES]. They proved the existence of a harmonic map in any homotopy class in the case where M and N are compact manifolds with $K_N \leqslant 0$. They deformed an arbitrary map through a nonlinear heat equation. By passing to the limit, with the appropriate estimates, one obtains a harmonic map in this way. In fact, harmonic maps are unique in their homotopy classes if $K_N < 0$ and rank $\geqslant 2$ [Hr]. Later, R. Hamilton [Ha] using the same method as in [ES] together with delicate estimates, settled the Dirichlet problem when M