3. Hopf fibrations with fibre S^3

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d) *There is no symmetry of* H *inducing the identity on the base space and reversing the orientations of the Hopf circles.*

Consider the symmetries $z \cdot Id$, $|z| = 1$, which multiply each coordinate in C^n by the complex number z of unit length. They induce the identity on the base space, and can be selected to take a fibre to itself by a preassigned rotation, proving a).

The transformations in $U(n)$ can take any complex line in C^n to any other by a preassigned orientation preserving rigid motion. Complex conjugation then adds the orientation reversing ones, proving b).

In particular, this implies c).

Suppose there were a symmetry of $H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$ taking each Hopf circle to itself with reversal of orientation. Then, by restriction to C^2, such a symmetry would also exist for $n = 2$. Its reversal of orientation on the total space S^3 would then contradict the remark following Proposition 2.1. This proves d). QED

Remarks. 1) Note that the existence of symmetries of H rotating each Hopf circle within itself shows again that these circles must be parallel.

2) Also note that a symmetry of $H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$ induces an isometry of the base space CP^{n-1} in its canonical metric. We remark without proof that *all* isometries of CP^{n-1} can be produced this way.

3. Hopf fibrations with fibre S^3

Choose orthonormal coordinates in R^{4n} and identify this space with quaternionic n-space H^n. A little care is needed in dealing with H^n because the quaternions form a *non*-commutative division algebra:

1) Scalars $v \in H$ will act on vectors $(u_1, ..., u_n) \in H^n$ from the *right*.

$$(u_1, ..., u_n)\, v = (u_1 v, ..., u_n v) .$$

2) H-linear transformations of H^n will be expressed by matrices of quaternions acting from the *left* (so as to commute with scalar multiplication).

The quaternionic lines in H^n, each looking like a real 4-plane, form the quaternionic projective space HP^{n-1} and fill out H^n, with any two meeting only at the origin. The unit 3-spheres on these quaternionic lines give us the *Hopf fibration*

$$H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1} .$$

PROPOSITION 3.1. *The Hopf 3-spheres on* S^{4n-1} *are parallel to one another.*

The proof is similar to that of Proposition 1.1 for Hopf circles; it uses the fact that scalar multiplication by i, j and k are isometries of H^n. Alternatively, it will follow, as in Remark 1 above, from Proposition 4.2 a.

QED

The Riemannian metric on HP^{n-1} which makes the Hopf projection $S^{4n-1} \to HP^{n-1}$ into a Riemannian submersion is known as the *canonical metric* on HP^{n-1}. The canonical metric on HP^1 makes it into a round 4-sphere of radius 1/2. This follows by the same argument given in Proposition 1.2 for the case $H: S^1 \hookrightarrow S^3 \to CP^1$.

4. Symmetries of the Hopf fibrations with fibre S^3

We now investigate the symmetries of the Hopf fibration

$$H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1}.$$

The symplectic group

$$Sp(n) = Gl(n, H) \cap O(4n)$$

consists of quaternionically linear maps which are also rigid. Since these maps take quaternionic lines to quaternionic lines, they must be symmetries of the above Hopf fibration.

There are other symmetries. For each unit quaternion v, consider the action of right scalar multiplication by v on H^n,

$$R_v(u_1, ..., u_n) = (u_1 v, ..., u_n v).$$

This map is certainly *not* H-linear, since

$$R_v[(u_1, ..., u_n)w] = (u_1 w v, ..., u_n w v),$$

while

$$[R_v(u_1, ..., u_n)] w = (u_1 v w, ..., u_n v w).$$

Nevertheless, R_v takes each quaternionic line in H^n to itself. Thus the group S^3 of unit quaternions, acting on H^n from the right, must also be counted among the symmetries of our Hopf fibration.

Since the symplectic group $Sp(n)$ acts on S^{4n-1} from the left, while the group S^3 of unit quaternions acts from the right, these two actions