1. Hopf fibrations with fibre S^1

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1. Hopf fibrations with fibre S^1

We describe the Hopf fibration

$$H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$$

as follows. Choose orthonormal coordinates in real $2n$-space R^{2n} and write

$$\begin{aligned}(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) &= (x_1 + ix_2, \ldots, x_{2n-1} + ix_{2n}) \\ &= (u_1, \ldots, u_n),\end{aligned}$$

thus identifying R^{2n} with complex n-space C^n.

The complex lines in C^n, each looking like a real 2-plane, form the complex projective space CP^{n-1} and fill out C^n, with any two meeting only at the origin. The unit circles on these complex lines give us the *Hopf fibration* of S^{2n-1}.

The simplest case occurs for $n = 2$. The complex lines in C^2 are of the form

$$L_m = \{(u, mu) : u \in C\} \quad \text{for each} \quad m \in C,$$

and

$$L_\infty = \{(0, v) : v \in C\}.$$

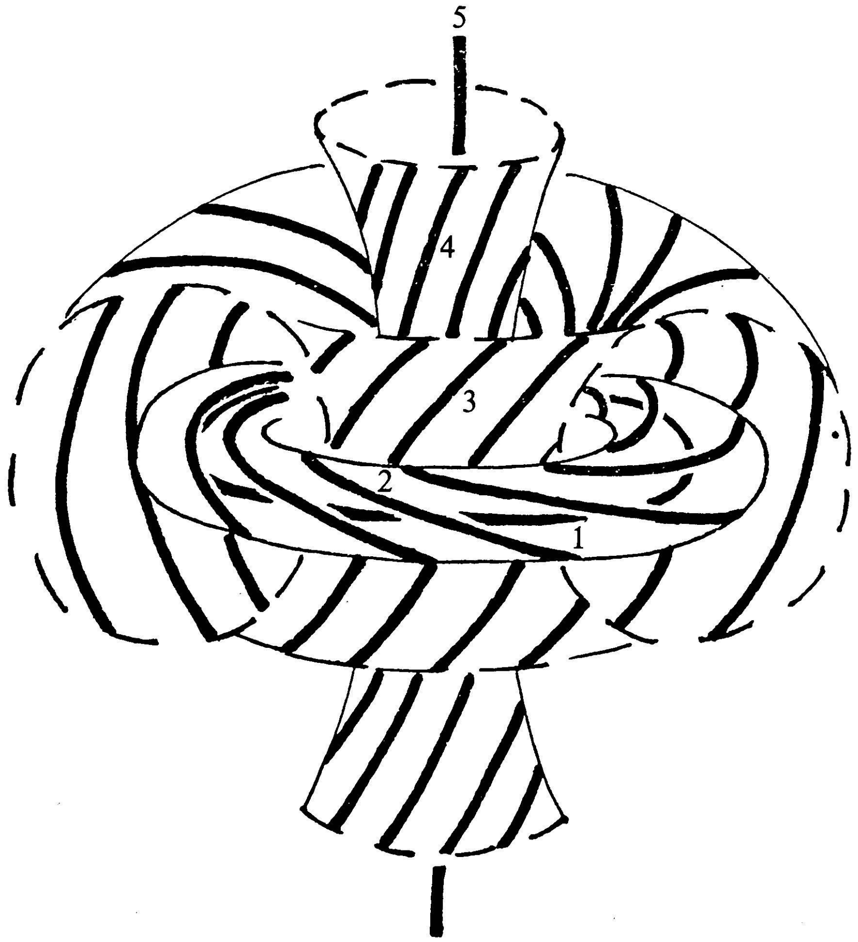

Figure 1

Note that there is one Hopf circle for each complex number m, and one for the number ∞. So the set of Hopf circles is topologically a 2-sphere.

Above is a sketch of the Hopf fibration $H: S^1 \hookrightarrow S^3 \to S^2$, due to Roger Penrose [Pe].

The portions of the this sketch may be identified as follows:

1) Circle $x_1^2 + x_2^2 = 1, \quad x_3 = 0, x_4 = 0$	fibre
2) Torus $x_1^2 + x_2^2 = 3/4,\ x_3^2 + x_4^2 = 1/4$	union of fibres
3) Torus $x_1^2 + x_2^2 = 1/2,\ x_3^2 + x_4^2 = 1/2$	union of fibres
4) Torus $x_1^2 + x_2^2 = 1/4,\ x_3^2 + x_4^2 = 3/4$	union of fibres
5) Circle $x_1 = 0, x_2 = 0,\ x_3^2 + x_4^2 = 1$	fibre

In the construction of the Hopf fibration of S^{2n-1} by great circles, we began by choosing orthonormal coordinates for R^{2n}. A different choice of such coordinates simply turns the picture of the Hopf fibration around by a rigid motion of S^{2n-1}, and we refer to all of these as "Hopf fibrations".

A key geometric feature of the Hopf fibrations is given by

PROPOSITION 1.1. *The Hopf circles on* S^{2n-1} *are parallel to one another.*

What do we even mean by this? Two subsets P and Q of a metric space will be said to be *parallel* if there is some real number d such that each point of P has minimum distance d from Q, and vice versa. If P and Q are parallel great circles on S^{2n-1} at distance d apart, then each lies on the boundary of a tubular neighborhood of radius d about the other.

To see this with more precision, first suppose that P and Q are arbitrary great circles on S^{2n-1}, and use the same symbols to denote the 2-planes through the origin that they span in R^{2n}. Let α_1 denote the smallest angle that any line in P makes with Q, and let α_2 denote the largest such angle. Then $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \pi/2$. These angles are called the *principal angles* between P and Q.

One can always choose an orthonormal basis $e_1, ..., e_{2n}$ for R^{2n} so that e_1 and e_2 form an orthonormal basis for P, while $\cos\alpha_1\, e_1 + \sin\alpha_1\, e_3$ and $\cos\alpha_2\, e_2 + \sin\alpha_2\, e_4$ form an orthonormal basis for Q. Then P and Q are parallel if and only if the two principal angles α_1 and α_2 are equal.

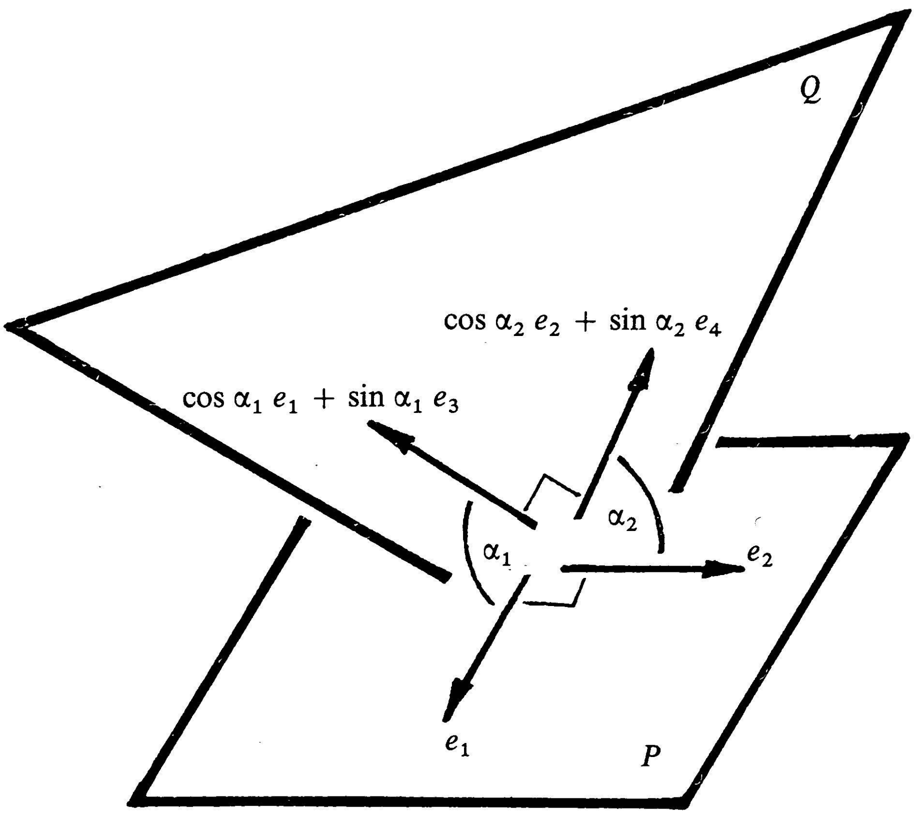

FIGURE 2

Note that with respect to these bases, the matrix for orthogonal projection of P onto Q (or vice versa) is given by

$$\begin{pmatrix} \cos \alpha_1 & 0 \\ 0 & \cos \alpha_2 \end{pmatrix}.$$

Thus P and Q are parallel if and only if orthogonal projection of P to Q is a conformal map. For future use, we also note that if A is the matrix of a linear map with respect to orthonormal bases, then that map is conformal if and only if $A\, A^t = \lambda\, I$.

To prove the proposition, let P and Q be two Hopf circles on S^{2n-1}. If u is any unit vector in the 2-plane P, then u and iu form an orthonormal basis for P. Likewise we get an orthonormal basis v and iv for Q. With respect to these bases, the matrix A of orthogonal projection of P onto Q is given by

$$\begin{pmatrix} a = \langle u, v \rangle & b = \langle u, iv \rangle \\ c = \langle iu, v \rangle & d = \langle iu, iv \rangle \end{pmatrix}.$$

But multiplication by i is an isometry; hence $a = d$ and $b = -c$. Thus

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{and} \quad A\,A^t = \begin{pmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix},$$

showing that A is conformal, and hence that P and Q are parallel. QED

We remark here that, unlike the usual situation in Euclidean space, being parallel is not a transitive relation in spherical geometry. Consider, for example, the following three great circles in S^3:

$$P_1 = \{(x, y, x, y) : x^2 + y^2 = 1/2\}$$
$$P_2 = \{(x, y, 0, 0) : x^2 + y^2 = 1\}$$
$$P_3 = \{(x, y, x, -y) : x^2 + y^2 = 1/2\}.$$

Then P_1 and P_3 are each parallel to P_2, but certainly not to each other, since they meet in two points.

Since the Hopf fibrations of S^{2n-1} have parallel fibres, they can be viewed as Riemannian submersions as follows.

Let $\pi: M \to N$ be a smooth map between smooth manifolds. This map is said to be a *submersion* if its differential π_* has maximal rank at each point. A submersion between closed manifolds must be a fibration.

If in addition M and N are Riemannian manifolds, then a submersion between them is said to be a *Riemannian submersion* if its differential preserves the lengths of tangent vectors orthogonal to the fibres $\pi^{-1}(y)$, $y \in N$.

Suppose now that $\pi: M \to N$ is a submersion of one complete connected smooth manifold onto another. The following facts are easy to deduce:

1) *If M and N have Riemannian metrics which make π a Riemannian submersion, then the fibres of π are parallel in M.*

2) *If M has a Riemannian metric in which the fibres of π are parallel, then one can choose a Riemannian metric on N in terms of which π becomes a Riemannian submersion.*

In particular, there is a Riemannian metric on CP^{n-1} which makes the Hopf projection $\pi: S^{2n-1} \to CP^{n-1}$ into a Riemannian submersion. This is known as the *canonical metric* on CP^{n-1}. The distance between points on CP^{n-1} equals the distance between corresponding Hopf fibres on S^{2n-1}.

PROPOSITION 1.2. *The canonical metric on CP^1 makes it into a round two-sphere of radius 1/2.*

We've already noted that for the lowest dimensional Hopf fibration $H: S^1 \hookrightarrow S^3 \to CP^1$, the base space is topologically a two-sphere. Let P denote one of the fibres of H, say the unit circle on the x_1x_2-plane. Let $P^\perp$ denote the orthogonal fibre, in this case the unit circle on the x_3x_4-plane. We let P correspond to the north pole and $P^\perp$ to the south pole on a round two-sphere $S^2(1/2)$ of radius 1/2.

For each quarter circle on S^3 from P to $P^\perp$, orthogonal to P and $P^\perp$, we obtain a family of fibres of H, one through each point of the quarter circle. These will correspond to the points of a semicircle on $S^2(1/2)$ from the north pole to the south pole.

Now consider all the fibres of H which are at distance α from P, $0 < \alpha < \pi/2$. They fill out the torus

$$\begin{aligned} T_\alpha &= \{x_1^2 + x_2^2 = \cos^2 \alpha,\ x_3^2 + x_4^2 = \sin^2 \alpha\} \\ &= S^1(\cos \alpha) \times S^1(\sin \alpha)\,. \end{aligned}$$

Every fibre on this torus is the graph of a "linear" bijection from $S^1(\cos \alpha)$ to $S^1(\sin \alpha)$. Each such fibre meets a small circle $(\cos \alpha, 0, 0, 0) \times S^1(\sin \alpha)$ at a single point. But these points are further apart than the actual distances between the fibres. The following diagram shows that in the limit, as q' approaches q, the scale correction factor is $\cos \alpha$.

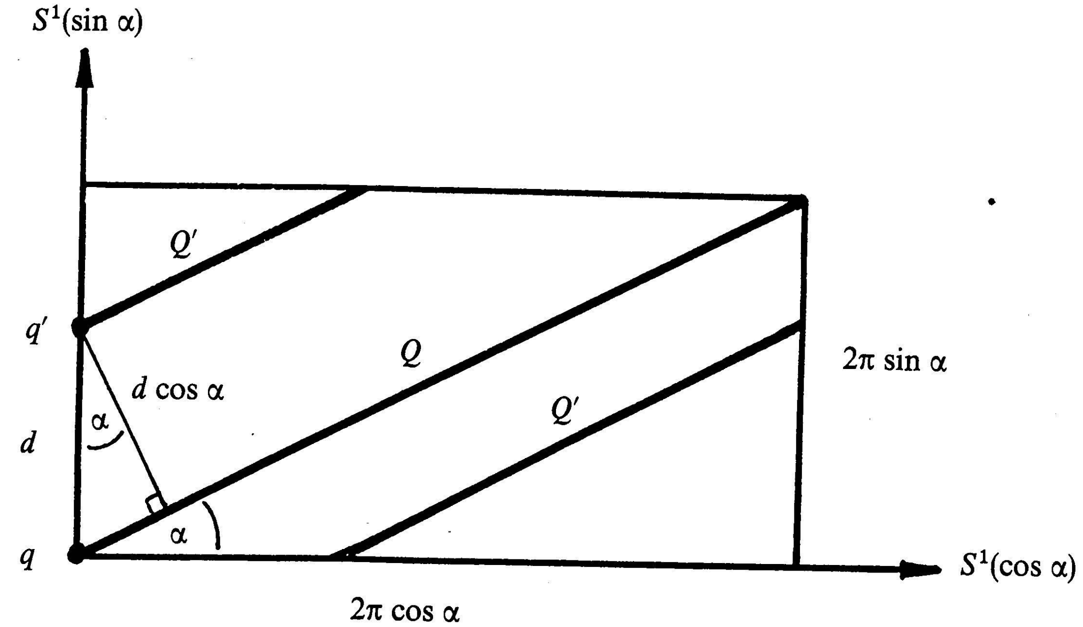

FIGURE 3

Hence the fibres of H which lie on the torus T_α form a circle of radius $\sin\alpha\cos\alpha$. But a circle of latitude on $S^2(1/2)$, located at distance α from the north pole, has radius $(1/2)\sin 2\alpha = \sin\alpha\cos\alpha$. It follows that there is a correspondence between fibres of H and points of $S^2(1/2)$ which is a Riemannian isometry, proving the proposition. QED

Besides being parallel, the fibres of the Hopf fibration are assembled in a very regular way. The following two geometric features give an expression of this regularity, and were important in [GWZ].

1) *Constancy Feature.* Refer again to the figure showing the Hopf fibration of S^3, in which we see S^3 decomposed into a pair of orthogonal great circles and a family of intermediating tori:

$$\begin{aligned} T_0 &= S^1(1) \times 0 \\ T_\alpha &= S^1(\cos\alpha) \times S^1(\sin\alpha) \qquad 0 < \alpha < \pi/2 \\ T_{\pi/2} &= 0 \times S^1(1)\,. \end{aligned}$$

Any two of these intermediating tori are a constant distance apart, and hence parallel to one another. There is a natural "radial projection" map between them, which matches closest neighbors on the two surfaces. It is easy to see that this map also matches Hopf circles, and in this sense we regard the Hopf fibration as "constant" on the family of tori. A corresponding phenomenon can be observed in all the Hopf fibrations.

2) *Inductive Feature.* A Hopf fibration contains within itself copies of lower dimensional Hopf fibrations, and can be regarded as assembled from these in a certain way. For example, just as C^n contains C^{n-1}, so does the Hopf fibration of S^{2n-1} contain the Hopf fibration of S^{2n-3}.

2. Symmetries of the Hopf fibrations with fibre S^1

Let $H: S^1 \hookrightarrow S^{2n-1} \to CP^{n-1}$ denote a Hopf fibration with fibre S^1. By a *symmetry* of H we mean a rigid motion of S^{2n-1} which takes Hopf circles to Hopf circles. We want to find these symmetries explicitly.

The unitary group

$$\begin{aligned} U(n) &= Gl(n, C) \cap O(2n) \\ &= \text{complex general linear group} \cap \text{orthogonal group} \end{aligned}$$