# §1. Geometry and combinatorics

Corollary 5 in § 1) which hold for reflection groups in general as was demonstrated by Davis [3] (see also [18]).

The proofs of these general properties in the literature depend on the particular type of the reflection groups considered. The best known proofs are those in [2] for the linear (or affine) reflection groups acting properly discontinuously on the whole space, and Davis [3] adapted them to the general case of topological reflection groups.

The purpose of this paper is to supply geometric proofs of the basic properties of general reflection groups as opposed to adapting the formal arguments of [2]. For simplicity of exposition we assume in the paper that $M$ is a differentiable (actually $C^1$) manifold and that the group action is $C^1$. Extension to the topological manifolds does not require new ideas and is left to the reader.

The rationale for this paper is twofold. First, the basic properties of reflection groups are stated (and proved) here in a form particularly useful for applications (cf. [5], [9], [11]). Second and more important, the simplicity of the geometric proofs presented here will make the subject more accessible to the general mathematical public.

In conclusion let me mention that reflection groups that do not act properly discontinuously are also useful (cf. [19], [4], [8]) but the results of the paper do not extend to them.

I would like to thank Mike Davis and the referee for pointing out an error in the original version of the paper. [1])

## § 1. Geometry and combinatorics

Throughout the paper $M$ is a connected differentiable manifold (possibly with boundary).

*Definition 1.* A reflection of $M$ is a diffeomorphism $s$ such that $s^2 = 1$ and the set $M_s$ of fixed points of $s$ has codimension 1. A reflection $s$ is called separating if $M \backslash M_s$ is disconnected. A reflection group $W$ acting on $M$ is a discrete group of diffeomorphisms of $M$ generated by separating reflections.

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[1]) I. N. Bernstein told me that E. B. Vinberg has an unpublished manuscript on reflection groups which is similar to this one.

LEMMA 1. *Let* $s$ *be a reflection of* $M$. *Then* $M \backslash M_s$ *has at most two connected components.*

*Proof.* Let $x_0, x_1 \in M\backslash M_s$ and let $x(t)$ be a continuous path joining them. We can assume without loss of generality that $x(t)$ is piecewise differentiable and that it intersects $M_s$ transversally. Let $x(t_1), ..., x(t_N)$ be the points of intersection. Consider the new path $\tilde{x}(t)$ where

$$\tilde{x}(t) = x(t), \quad 0 \leqslant t \leqslant t_1, \quad \tilde{x}(t) = sx(t), \quad t_1 \leqslant t \leqslant t_2,$$
$$\tilde{x}(t) = x(t), \quad t_2 \leqslant t \leqslant t_3, \text{ etc.}$$

(see fig. 1). Deform the path $\tilde{x}(t)$ slightly in small neighborhoods of $x(t_1), ..., x(t_N)$ to make it come off $M_s$ (if $\tilde{x}(t)$ does not cross $M_s$ at $t_i$).

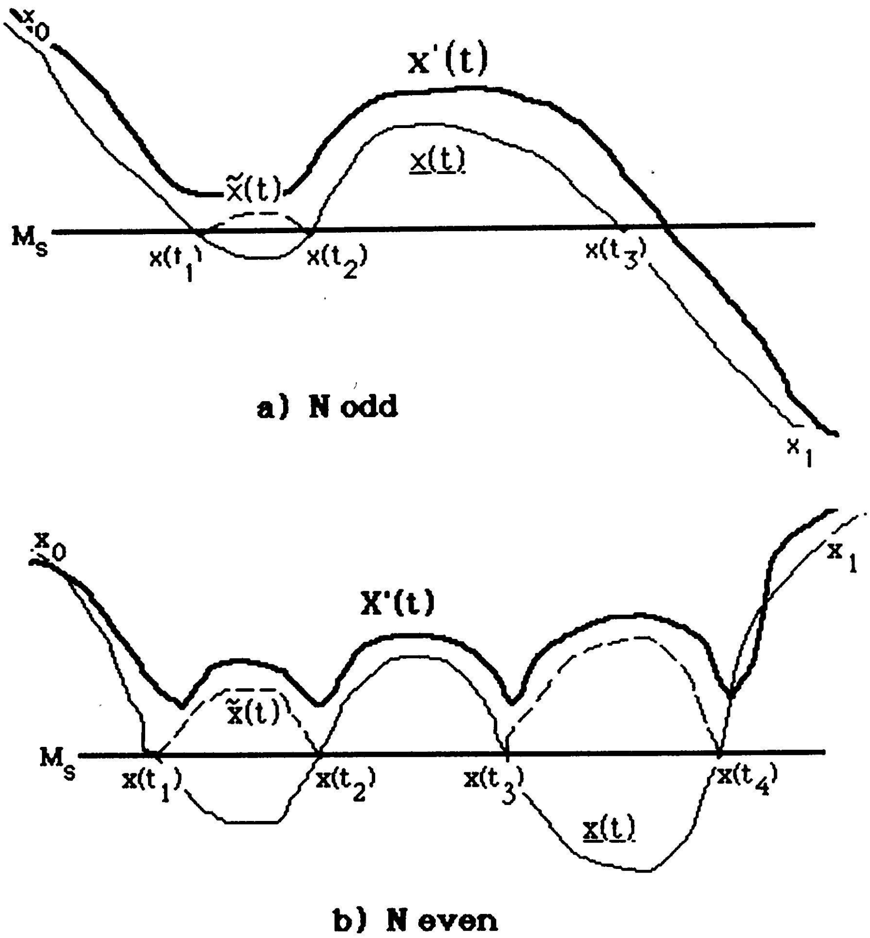


FIGURE 1
$M\backslash M_s$ has at most two connected components

The resulting path $x'(t)$ does not intersect $M_s$ at all if $N$ is even and intersects $M_s$ only at $x(t_N)$ is $N$ is odd. Thus any $x, y \in M\backslash M_s$ can be joined by a continuous path intersecting $M_s$ at most once.

Assume that $M\backslash M_s$ has three connected components $X$, $Y$, $Z$ and choose points $x, y, z$ in $X$, $Y$, $Z$ respectively. Then there are paths $\gamma, \tilde{\gamma}$ from $x$ to $y$

and from $y$ to $z$ respectively intersecting $M_s$ once. The path $\tilde{\gamma}\gamma$ goes from $x$ to $z$ and intersects $M_s$ twice. By previous argument we can find another path $\gamma'$ from $x$ to $z$ that does not intersect $M_s$ at all. This contradiction proves the Lemma.

COROLLARY 0. *Let* $M \backslash M_s$ *be disconnected and let* $x, y \in M \backslash M_s$. *Let* $\gamma$ *be a continuous path in* $M$ *joining* $x$ *with* $y$ *and intersecting* $M_s$ *transversally. Then* $x, y$ *belong to the same component of* $M \backslash M_s$ *if and only if* $\gamma$ *intersects* $M_s$ *an even number of times.*

*Proof.* Let $\gamma$ intersect $M_s$ $N$ times. By the argument of Lemma 1, we can find another path $\gamma'$ from $x$ to $y$ that intersects $M_s$ once if $N$ is odd and does not intersect $M_s$ if $N$ is even. Thus it suffices to prove that if $\gamma$ joining $x$ with $y$ intersects $M_s$ once then $x$ and $y$ belong to different connected components of $M \backslash M_s$. Assume the opposite and let $z$ belong to the other component. Then there is a path $\tilde{\gamma}$ from $y$ to $z$ intersecting $M_s$ once. The composition $\tilde{\gamma}\gamma$ joints points in different connected components of $M \backslash M_s$ and intersects $M_s$ twice. By the argument of Lemma 1, this is impossible.

Denote by $M/s$ the quotient of $M$ by the action of $s$ endowed with the natural topology.

PROPOSITION 1. *Let* $s$ *be a reflection of* $M$.

(i) *Assume that* $s$ *is separating. Then* $M$ *is orientable if and only if* $M/s$ *is.*

(ii) *Assume that* $s$ *is not separating. If* $M$ *is orientable then* $M/s$ *is not orientable.*

*Proof.* Let $X$ be a connected manifold with the boundary $\partial X \neq \phi$. Define the doubling $dX$ of $X$ as the manifold obtained by gluing two copies of $X$ along the common boundary. Clearly $dX$ is orientable if and only if $X$ is.

Let $M_s$ separate $M$ and let $X_0$, $Y_0$ be the connected components of $M \backslash M_s$. Let $X = X_0 \bigcup M_s$, $Y = Y_0 \bigcup M_s$ be their closures in $M$. Then $s: X \to Y$ is a diffeomorphism which identifies $X$, $Y$ with $M/s$ and $M$ with $dX$. This proves (i).

(ii) Let $x \in M$ be sufficiently close to $M_s$. Then $x$ and $sx$ belong to the same open ball in $M$ and we orient the tangent spaces at $x$ and $sx$ simultaneously. Let $\gamma$ be a continuous path in $M \backslash M_s$ from $x$ to $sx$. Since $M$

is orientable, moving along $\gamma$ does not change the orientation. Since $s$ reverses the orientation, moving along the loop $p\gamma$ in the quotient $p: M \to M/s$ we come back to $px$ with the orientation reversed.

*Examples.* 1. Let $M = S^1 \times S^1$ be the twodimensional torus and let $s$ be the reflection about the diagonal (see fig. 2, a)). Then $M/s$ is the Moebius band.

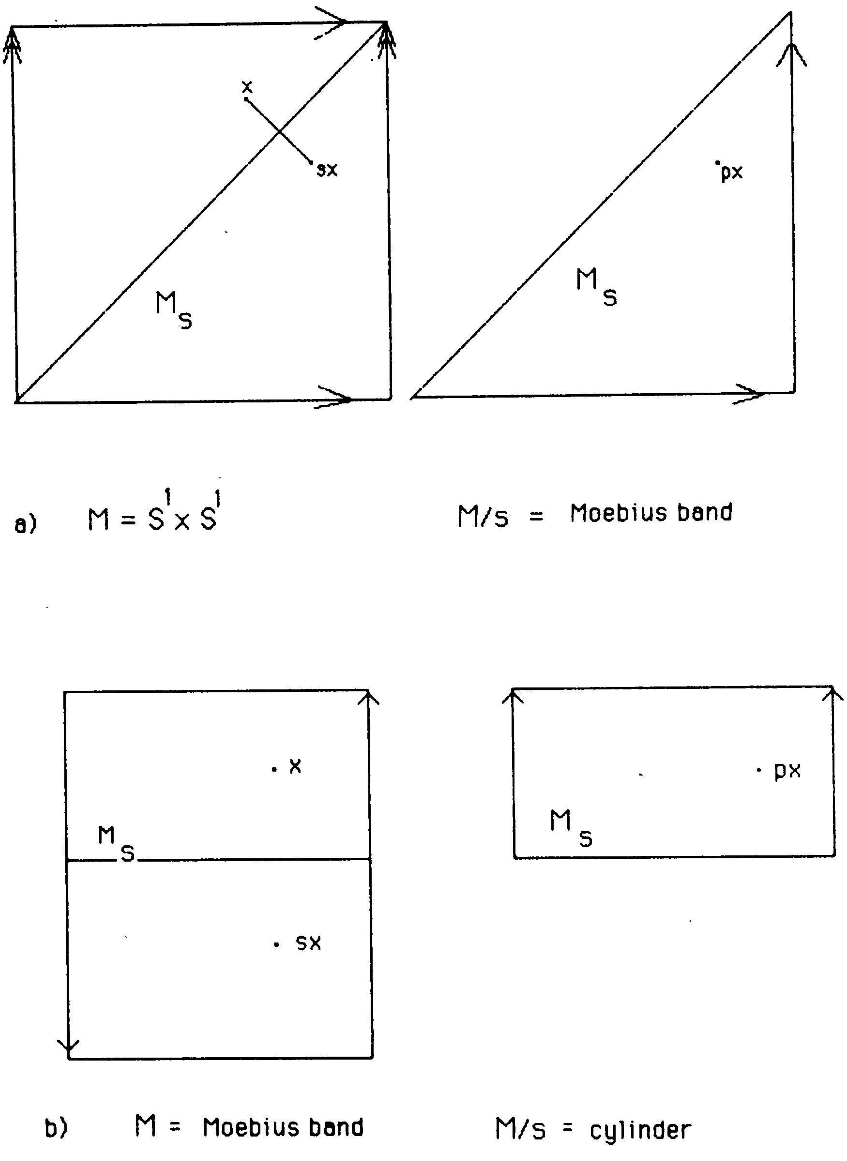


FIGURE 2

2. Let $s$ be the reflection of the Moebius band $M$ about the midline (see fig. 2, b)). Then $M/s$ is the cylinder.

*Remark.* Proposition 1, (ii) shows that if $s$ is not separating then $M$ and $M/s$ can not be both orientable. The following example shows that $M$ and $M/s$ can be both nonorientable.

3. Let $M$ be the product of two Moebius bands and let $s$ be the product of the reflection in midline (Example 2) and the identity map. Then $M/s$ is the product of the cylinder and the Moebius band. Thus $s$ does not separate $M$ and both $M$ and $M/s$ are not orientable.

COROLLARY 1. *If $M$ is simply connected then any reflection $s$ of $M$ is separating.*

*Proof.* Since $s$ has fixed points, $M/s$ is simply connected, thus orientable. If $s$ is not separating then, by Proposition 1, (ii), $M$ is not orientable, contrary to the assumption.

In the rest of the paper we consider only separating reflections and groups generated by them. By Corollary 1, if $M$ is simply connected (which holds in many applications) then the assumption is automatically satisfied.

Let us establish some terminology. The closures $M_s^\varepsilon$, $\varepsilon = \pm 1$, of connected components of $M \backslash M_s$ are the two halfspaces corresponding to $s$. If $A \subset M$ intersects only one connected component of $M \backslash M_s$ we denote the corresponding halfspace by $M_s(A)^+$ and the other one by $M_s(A)^-$.

Let $W$ be a reflection group acting on $M$ and let $R \subset W$ be the set of reflections in $W$. The sets $M_s, s \in R$ are called the (reflecting) walls of $M$ and the closures of connected components of $M_{\text{reg}} = M \backslash \bigcup_{s \in R} M_s$ are the chambers of $M$. $M_{\text{reg}}$ is the set of regular points of $M$. Since a wall $M_s$ defines $s$ uniquely, we identify $R$ with the set of walls of $M$. Points $x \in M$ that belong to no more than one wall are the semiregular points of $M$. The walls of a chamber $C$ are such $M_s$ that $\dim (M_s \cap C) = n-1$, their intersections with $C$ are the faces of $C$. Walls of $C$ correspond to a subset $S_C \subset R$. Nonempty intersections of faces of $C$ are the facets of $C$.

Two chambers $C \neq D$ are adjacent if they have a common face. Let $M_r$ be the unique wall containing this face, then $D = rC$. A sequence $C_0, C_1, ..., C_N$ of chambers is a gallery (of length $N$, going from $C_0$ to $C_N$) if for $i = 1, ..., N$ the chambers $C_{i-1}$ and $C_i$ are adjacent. The sequence $(r_1, ..., r_N)$ of reflections defined by $C_i = r_i C_{i-1}$ is called the reflection sequence corresponding to the gallery $C_0, ..., C_N$. A gallery $C_0, ..., C_N$ crosses $M_r$ if $r$ is contained in the corresponding sequence $(r_1, ..., r_N)$. A minimal gallery going from $C$ to $D$ is a gallery of minimal length which is by definition the distance $d(C, D)$ between $C$ and $D$. A wall $M_r$ separates chambers $C$ and $D$ if $C \subset M_r^\varepsilon$ and $D \subset M_r^{-\varepsilon}$. Denote by $R(C, D) \subset R$ the set of walls separating $C$ from $D$. The group $W$ acts on $R$ by conjugations $r \to grg^{-1}$ which we denote for brevity by $g \cdot r$. Then $gM_r^\varepsilon = M_{g \cdot r}^\delta$, $\varepsilon$, $\delta = \pm 1$.

PROPOSITION 2. *Let* $C = C_0, C_1, ..., C_N = D$ *be a gallery and let* $(r_1, ..., r_N)$ *be the corresponding sequence of reflections.*

(i) *The set of reflections* $r$ *contained in* $(r_1, ..., r_N)$ *an odd number of times is* $R(C, D)$.

(ii) *The following assertions are equivalent:*

a) *gallery* $C_0, ..., C_N$ *is minimal;*

b) $d(C_i, C_j) = |i - j|$ *for any* $i, j = 0, ..., N$;

c) *there are no repetitions in the sequence* $(r_1, ..., r_N)$.

*Proof.* A differentiable path $\{x(t): 0 \leqslant t \leqslant 1\}$ on $M$ is called regular if for all but a finite number $0 < t_1 < ... < t_N < 1$ of moments of time $x(t)$ is regular, $x(t_i)$ is semiregular for $i = 1, ..., N$ and the curve $x(t)$ is transversal to the set $\bigcup_{s \in R} M_s$. Then for $t \neq t_1, ..., t_N$ $x(t)$ belongs to a unique chamber and the sequence $C_0, ..., C_N$ thus defined is a gallery with $t_i$

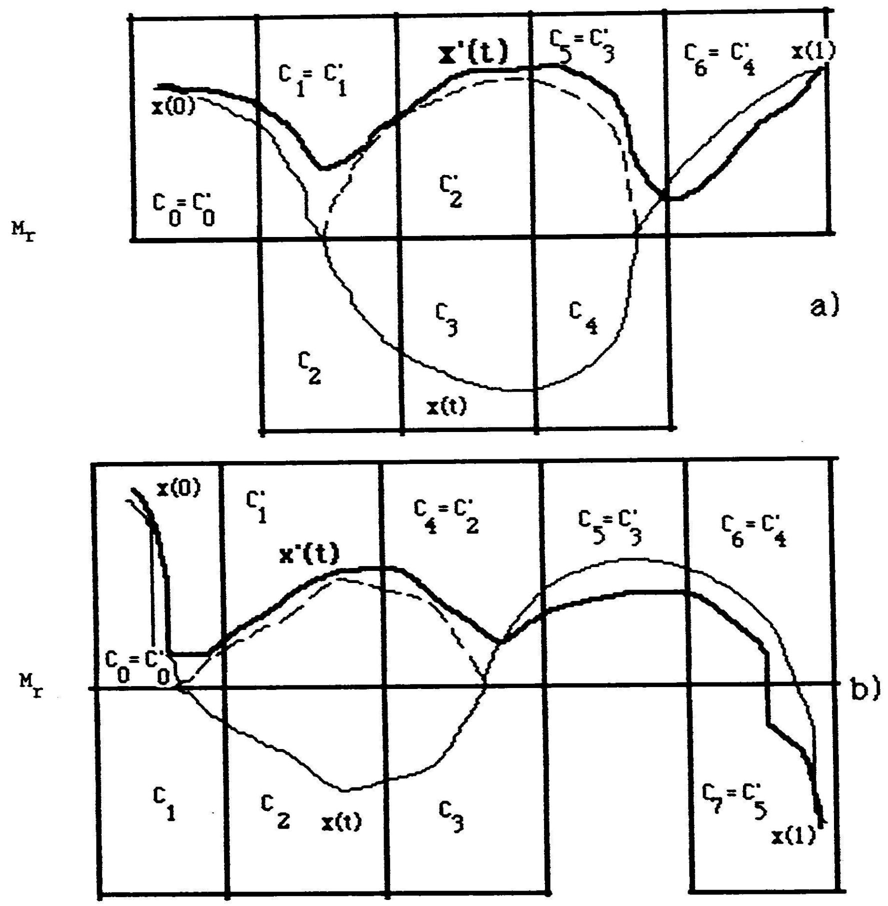


FIGURE 3
To the proof of Proposition 2. Galleries and paths along them

being the moment of time when $x(t)$ crosses from $C_{i-1}$ to $C_i$. We say that $C_0, ..., C_N$ is the gallery along the path $\{x(t)\}$. Given a gallery $C_0, ..., C_N$ there is always a regular path $\{x(t)\}$ such that $C_0, ..., C_N$ is the gallery along it. We say that $\{x(t)\}$ goes along the gallery.

Let $C_0, ..., C_N$ be a gallery going from $C = C_0$ to $D = C_N$ and let $\{x(t)\}$ be a regular path along it. For any $r \in R$ let $t_{i_1} < ... < t_{i_{N(r)}}$ be the consecutive moments of time when $\{x(t)\}$ intersects $M_r$. By Corollary 0, $N(r)$ is even (resp. odd) if and only if $r \notin R(C, D)$ (resp. $r \in R(C, D)$) which proves (i). Assertions a) and b) of (ii) are obviously equivalent. By (i) every $r \in R(C, D)$ is contained in $(r_1, ..., r_N)$ at least once, thus $N \geqslant | R(C, D) |$. Assume that $N > | R(C, D) |$. Then either there is $r \notin R(C, D)$ that occurs in $(r_1, ..., r_N)$ (necessarily an even number of times) or there is $r \in R(C, D)$ that occurs in $(r_1, ..., r_N)$ more than once. Assume the first possibility and let $r$ occur $2m$ times in $(r_1, ..., r_N)$. Using the proof of Lemma 1 we construct a new regular path $\{x'(t)\}$ which joins $x(0)$ with $x(1)$ and does not cross $M_r$ at all (see fig. 3, a)). The gallery along $\{x'(t)\}$ has $N + 1 - 2m$ chambers and does not cross $M_r$. Analogous argument shows that if there is $r \in R(C, D)$ that occurs $2m + 1 > 1$ times in $(r_1, ..., r_N)$ then there is a new gallery from $C$ to $D$ which is by $2m$ shorter then $C_0, ..., C_N$ and crosses $M_r$ once (see fig. 3, b)). Thus the sequence $(r_1, ..., r_N)$ corresponding to a minimal gallery can contain only $r \in R(C, D)$ and no more than once. On the other hand by (i), it must contain every $r \in R(C, D)$ at least once. This proves the Proposition and the following.

COROLLARY 2. 1) *A gallery* $C = C_0, ..., C_N = D$ *is minimal if and only if* $N = | R(C, D) |$. *Thus* $d(C, D) = | R(C, D) |$. 2) *For any gallery* $C = C_0, ..., C_N = D, (-)^N = (-)^{d(C, D)}$.

COROLLARY 3. *Let* $D \neq C$ *be two chambers, let* $M_s$ *(resp.* $M_r$*) be a wall of* $C$ *(resp.* $D$*) such that* $r, s \in R(C, D)$. *Then there exists a minimal gallery* $C_0 = C, ..., C_N = D$ *such that* $C_1 = sC$ *and* $C_{N-1} = rD$.

*Proof.* If $d(C, D) = 1$ then $r = s$ and the assertion is trivial. If $t \in R$ and $t \neq s$ then $t$ cannot separate $sC$ from $C$. If besides $t \in R(C, D)$ then $C$, $sC \subset M_t(D)^-$ and if $t \notin R(C, D)$ then $C$, $sC \subset M_t(D)^+$. Therefore $R(sC, D) = R(C, D)\backslash\{s\}$ and $d(sC, D) = d(C, D) - 1$. This proves the assertion by induction on $d(C, D)$.

The group $W$ naturally acts on the set of chambers of $M$. Choose one chamber $C_+$ to be the fundamental chamber and let $S = S_{C_+}$ be the set of reflections in the walls of $C_+$. Elements $s \in S$ are called simple reflections.

PROPOSITION 3.

(i) *$W$ acts simply transitively on the set of chambers.*

(ii) *$S$ generates $W$.*

(iii) *Any $r \in R$ is conjugate to some $s \in S$.*

(iv) *Let $g \in W$ and let $g = s_1 \dots s_N$ be a decomposition of $g$ into simple reflections. Then the sequence*

$$C_0 = C_+, C_1 = s_1 C_+, \dots, C_i = s_1 \dots s_i C_+, \dots, C_N = s_1 \dots s_N C_+$$

*is a gallery. This establishes a one to one correspondence between the words in $s_i$ and galleries starting from $C_+$.*

*Proof.* Denote $d(C_+, C)$ by $d(C)$ and $R(C_+, C)$ by $R(C)$. Let $\tilde{W}$ be the subgroup of $W$ generated by $S$. We have seen in the proof of Corollary 3 that if $d(C) > 0$ then there is $r \in R(C)$ such that $d(rC) = d(C) - 1$. Assuming that $rC = wC_+$ for some $w \in \tilde{W}$ we have $r = wsw^{-1}$ for some $s \in S$, thus $r \in \tilde{W}$ and $C = r \cdot rC = rwC_+$ where $rw \in \tilde{W}$. This proves by induction on $d(C)$ that $\tilde{W}$ acts transitively on the set of chambers.

Let $r \in R$ and let $C$ be such that $M_r$ is a wall of $C$. Then there is $w \in \tilde{W}$ such that $w^{-1}C = C_+$ thus $w^{-1}M_r$ is a wall of $C_+$, that is $w^{-1}M_r = M_s$ for some $s \in S$, therefore $r = wsw^{-1}$ which shows that $R \subset \tilde{W}$ and proves (iii). The group $W$ is generated by $R$ and $R \subset \tilde{W}$ thus $W = \tilde{W}$ which proves (ii). Let $C_i, i = 0, \dots, N$ be the sequence of chambers defined in (iv). Since

$$C_{i+1} = (s_1 \dots s_i) s_{i+1} (s_i \dots s_1) s_1 \dots s_i C_+ = r_{i+1} C_i$$

and since $r_{i+1} = (s_1 \dots s_i) s_{i+1} (s_1 \dots s_i)^{-1} \in S_{C_i}$ the chambers $C_{i+1}$ and $C_i$ are adjacent, thus $C_0, \dots, C_N$ is a gallery going from $C_+$ to $gC_+$. Let $C_0 = C_+, \dots, C_N$ be any gallery and let $(r_1, \dots, r_N)$ be the corresponding sequence of reflections. Set $g_i = r_i \dots r_1$ $i = 1, \dots, N$. Then $C_i = g_i C_+$ and $g_i^{-1} r_{i+1} g_i \in S$ for $i = 1, \dots, N$. Denote $g_i^{-1} r_{i+1} g_i$ by $s_{i+1}$. Then $g_{i+1} = r_{i+1} g_i = g_i s_{i+1}$ which shows by induction that $g_i = s_1 \dots s_i$ for $i = 1, \dots, N$. Thus the gallery $C_0, \dots, C_N$ corresponds to the word $s_1 \dots s_{i+1}$ which proves (iv). In particular, two words $s_1 \dots s_N$ and $s'_1 \dots s'_M$ represent the same $g \in W$ if and only if the corresponding galleries $C_+ = C_0, \dots, C_N$ and $C_+ -$

$= C'_0, \ldots, C'_M$ lead to he same chamber $gC_+ = C_N = C'_M$. Thus the mapping $g \to gC_+$ is one to one which proves (i).

Choose a fundamental chamber $C_+$ and let $S$ be the corresponding set of simple reflections generating $W$. A decomposition $g = s_1 \ldots s_N, s_i \in S$ of $g \in W$ is called minimal if it is the shortest possible. Then $N = d(g)$ is the length of $g$ and the distance $d(g, h)$ is defined by $d(g, h) = d(g^{-1}h)$. Denote $gC_+$ by $C_g$, $R(C_g)$ by $R(g)$ and $M_r(C_+)^{\pm}$ by $M_r^{\pm}$ respectively. Identify the set of halfspaces $M_r^{\pm}$ with $\hat{R} = R \times \{\pm 1\}$ and call elements $(r, \varepsilon) = \hat{r} \in \hat{R}$ the oriented reflections. A gallery $C_0, \ldots, C_N$ defines a sequence $(\hat{r}_1, \ldots, \hat{r}_N)$ of oriented reflections by

$$\text{(1)} \qquad \hat{r}_i = \begin{cases} (r_i, +1) & \text{if} \quad C_i \subset M_r^+ \\ (r_i, -1) & \text{if} \quad C_i \subset M_r^- \end{cases}$$

Denote by $\hat{r} \to g\hat{r}$ the action of $W$ on $\hat{R}$ corresponding to the natural action of $W$ on the set of halfspaces. Define a function $sgn: W \times R \to \{\pm 1\}$ by

$$\text{(2)} \qquad sgn(g, r) = \begin{cases} -1 & g \cdot r \in R(g) \\ 1 & g \cdot r \notin R(g) \end{cases}$$

and for $\hat{r} = (r, \varepsilon)$ set $-\hat{r} = (r, -\varepsilon)$.

Corollary 4.

(i) *For any* $g, h \in W$, $d(g, h) = d(C_g, C_h)$ *and* $d(g) = |R(g)|$.

(ii) $R(g) = \{r \in R: g^{-1}M_r^{\varepsilon} = M_{g^{-1}\cdot r}^{-\varepsilon}\}$.

(iii) *The action of* $W$ *on* $\hat{R}$ *is given by* $g(r, \varepsilon) = (g \cdot r, sgn(g, r)\varepsilon)$.

*Proof.* (i) follows from Proposition 3 and Corollary 2. Recall that $R(g) = \{r \in R: C_g \subset M_r^-\}$. Since $g^{-1}C_g = C_+$ we have $g^{-1}M_r^- = M_{g^{-1}r}^+$. On the other hand if $r \notin R(g)$ then $C_g \subset M_r^+$ therefore $g^{-1}M_r^+ = M_{g^{-1}r}^+$ which proves (ii).

(ii) is equivalent to the assertion that $gM_r^{\varepsilon} = M_{g\cdot r}^{-\varepsilon}$ if $g \cdot r \in R(g)$ and $gM_r^{\varepsilon} = M_{g\cdot r}^{\varepsilon}$ if $g \cdot r \notin R(g)$ which proves (iii).

For $x \in M$ denote by $W_x \subset W$ the isotropy subgroup of $x$ and by $R_x \subset R$ the set of $r \in R$ such that $rx = x$.

PROPOSITION 4.

(i) *Let* $x, y \in C$, $g \in W$ *and let* $gx = y$. *Then* $x = y$ *and* $g \in W_x$.

(ii) *For any* $x \in M$ *the group* $W_x$ *is generated by reflections* $r \in R_x$.

*Proof.* Let $C, D$ be such chambers that $C \cap D \neq \phi$. Since any wall $M_r \in R(C, D)$ separates $C$ from $D$, it contains $C \cap D$. A minimal gallery $C = C_0, ..., C_N = D$ going from $C$ to $D$ crosses only the walls $M_r \in R(C, D)$, thus every chamber $C_0, ..., C_N$ contains $C \cap D$ and reflections of the corresponding sequence $(r_1, ..., r_N)$ leave $C \cap D$ fixed pointwise. In the notation of (i), $y \in gC \cap C \neq \phi$. A minimal decomposition $g = s_1 ... s_N$ corresponds to a minimal gallery going from $C$ to $gC$ and $g = r_N ... r_1$. Thus $g$ leaves $C \cap gC$ pointwise fixed, so $y = x$. For $x \in M$ let $C$ be a chamber containing $x$. By the same argument as above any $g \in W_x$ is a product of $r_i \in R_x$ which proves (ii).

COROLLARY 5. *The natural mapping* $\varphi: C_+ \to M/W$ *is an isomorphism.*

*Proof.* By Proposition 3, (i), $\varphi$ is onto. By Proposition 4, (i) $\varphi$ is one to one.

For $r, s \in R$ denote by $m(s, r) \in \{1, ..., \infty\}$ the order of $rs$. Since $s^2 = 1$ for any $s \in R$ we have

$$\text{(3)} \quad \begin{aligned} &\text{(i)} && m(s, s) = 1 \\ &\text{(ii)} && m(r, s) = m(s, r) \geqslant 2 \quad \text{for} \quad r \neq s\,. \end{aligned}$$

*Definition 2* (cf. Bourbaki [2]). A Coxeter group is a group $W$ with a finite set $S$ of generators and a presentation

$$\text{(4)} \qquad W = \langle S: (sr)^{m(s, r)} = 1, \quad r, s \in S \rangle$$

where the function $m: S \times S \to \{1, ..., \infty\}$ satisfies (i) and (ii) above.

THEOREM 1. *Let* $W$ *be a reflection group acting on* $M$, *let* $C_+$ *be a fundamental chamber, let* $S \subset R$ *be the corresponding set of simple reflections and for* $s, r \in S$ *let* $m(s, r)$ *be the order of* $sr$. *Then* $W$ *is a Coxeter group with the presentation*

$$\text{(5)} \qquad W = \langle S: (sr)^{m(s, r)} = 1 \rangle\,.$$

*Proof.* If $C_0, ..., C_N$ and $C'_0, ..., C'_M$ are two galleries such that $C'_0 = C_N$ we define their product by $C_0, ..., C_N, C'_1, ..., C'_M$. The inverse of the gallery

$C_0, \ldots, C_N$ is by definition $C_N, \ldots, C_0$. A loop is a closed gallery $C_0, \ldots, C_N = C_0$. Any chamber of a loop can be taken for the starting chamber. If there are two loops passing through the same chamber $C_0$, we define their product based on $C_0$ in an obvious way.

The dihedral group $D_m$ is the Coxeter group of order $2m$ with the presentation

$$D_m = \langle s, r : s^2 = r^2 = (sr)^m = 1 \rangle . \tag{6}$$

It is isomorphic to the reflection group on $\mathbf{R}^2$ generated by reflections $s, r$ in two lines meeting at the angle $\pi/m$.

By Proposition 3, (iv), there is a one to one correspondence between relations $s_1 \ldots s_N = 1$, $s_i \in S$ and loops starting from $C_+$.

If $r$, $s$ are reflections in the walls of a chamber $C$, the group they generate is the dihedral group $D_{m(s,r)}$ and the defining relation $(rs)^{m(r,s)} = 1$ corresponds to the loop on $\mathbf{R}^2$ starting at $C$ and going around the origin visiting every chamber once (see fig. 4). Let us call such loops elementary and let us call loops of the form $C_0, \ldots, C_{N-1}, C_N, C_{N-1}, \ldots, C_0$ trivial.

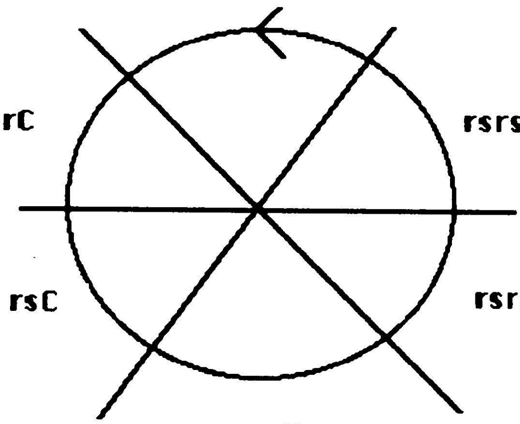


FIGURE 4
Loop corresponding to the relation $(rs)^3 = 1$

The statement of the Theorem is equivalent to the assertion that every loop is a product of elementary loops and trivial loops.

Let $C_0, \ldots, C_N$ be any gallery and set $d(C) = d(C, C_0)$. Then $d(C_{i+1}) = d(C_i) \pm 1$, $i = 0, \ldots, N-1$ and if $C_N = C_0$, the graph of the function $d : i \to d(C_i)$ looks like graphs on fig. 5, a), b).

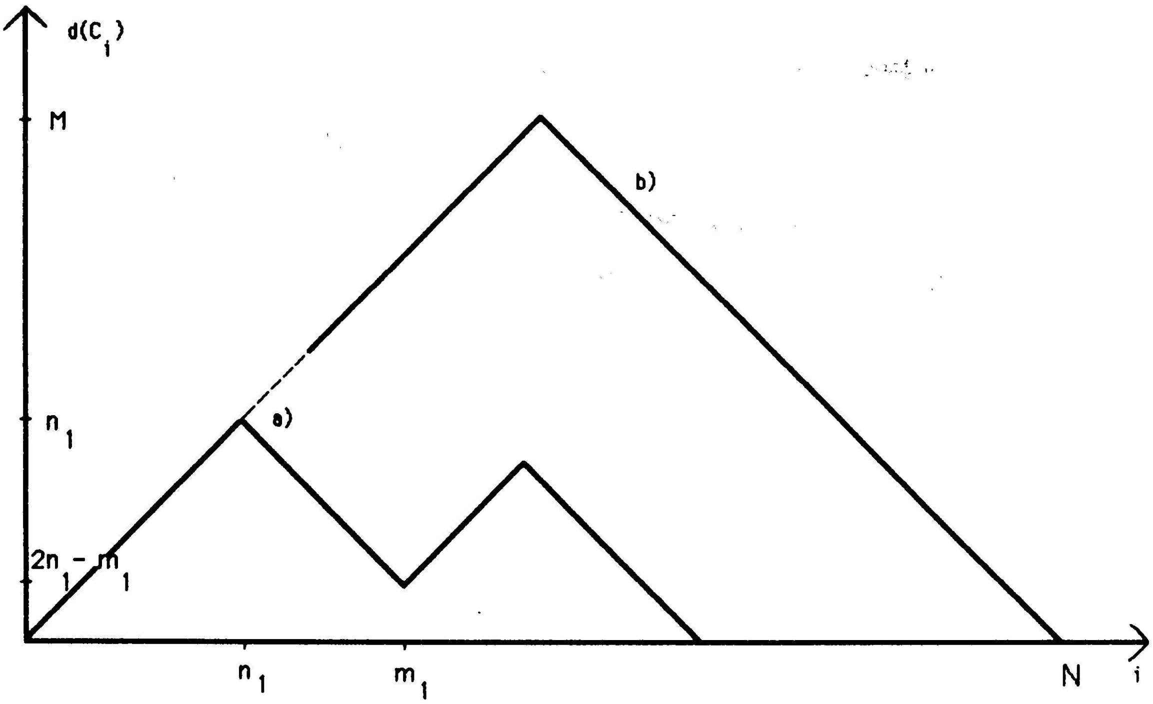


FIGURE 5 a), b)
Length function on loops

We call a loop perfect if it can not be decomposed into a product of shorter loops. It suffices to prove the assertion for perfect loops.

Assume that the function $d$ has more than one local maximum. Thus $d(C_i)$ increases until $i = n_1$, $d(n_1) = n_1$, then decreases to a local minimum at $i = m_1 > n_1$, $d(m_1) = n_1 - (m_1 - n_1) = 2n_1 - m_1$, then starts increasing again. Let $C_0, C'_1, ..., C'_{2n_1-m_1} = C_{m_1}$ be a minimal gallery going from $C_0$ to $C_{m_1}$. Then the original loop is the product of two loops

$$C_0, ..., C_{n_1}, ..., C_{m_1} = C_{2n_1-m_1}, C'_{2n_1-m_1-1}, ..., C'_1, C_0$$

and

$$C_0, C'_1, ..., C'_{2n_1-m_1} = C_{m_1}, C_{m_1+1}, ..., C_N = C_0 .$$

Each of them is shorter than the original one. Indeed, since the length of a loop is even, $N = 2M$ and $n_1 < M$ (see fig. 4, a)). The length of the first loop is $2n_1 < 2M$ and the length of the second is $(2n_1 - m_1) + (N - m_1) = N + 2(n_1 - m_1) < N$.

Thus the length function on a perfect loop must have a graph like one on fig. 4, b) no matter which chamber is used as a starting chamber.

Let $C_0, ..., C_{2m} = C_0$ be a perfect loop and let $(r_1, ..., r_{2m})$ be the corresponding reflection sequence. Since every subgallery of length $m$ is minimal, any reflection $r$ that occurs in $(r_1, ..., r_{2m})$ occurs twice. Moreover these $m$ distinct reflections must occur in the order $r_1, ..., r_m, r_1, ..., r_m$ (see Proposition 2, (ii)). It is convenient to arrange the sequence $r_1, ..., r_m, r_1, ..., r_m$ as a circle (see fig. 6). Then it becomes clear that it does not matter which chamber is taken for the starting chamber and that a half of the sequence determines the other half.

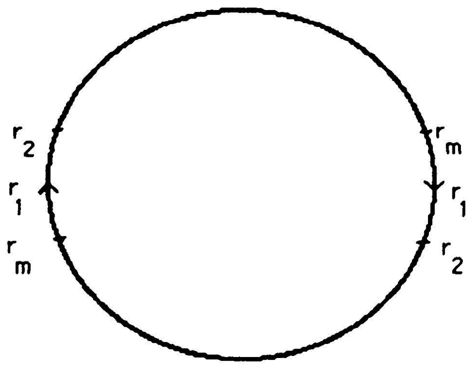


FIGURE 6
Reflection sequence of a perfect loop

The relation corresponding to a perfect loop has the form

(6) $$(r_m \dots r_1)^2 = 1\,.$$

Let $C_0, ..., C_n, n \leqslant m$ be a subgallery of a perfect loop and assume that there is another minimal gallery $C_0, C'_1, ..., C'_{n-1}, C_n$ going from $C_0$ to $C_n$ (and different from $C_0, C_{2m-1}, ..., C_{m+1}, C_m$ if $n = m$). Our loop is then the product of two loops

$$C_0, ..., C_n, C'_{n-1}, ..., C'_1, C_0 \quad \text{and} \quad C_0, C'_1, ..., C'_{n-1}, C_n, C_{n+1}, ..., C_0\,.$$

In the reflection sequence corresponding to the second loop the last $m$ reflections are $r_1, ..., r_m$ but the first $m$ are not $r_1, ..., r_m$. So it is not perfect therefore the original loop was not perfect.

The argument above shows that every subgallery $C_0, ..., C_{m-1}$ of length $m - 1$ of a perfect loop of length $2m$ is unique, i.e. there is no other minimal gallery going from $C_0$ to $C_{m-1}$ and the only other minimal gallery leading from $C_0$ to $C_m$ is the other half of the loop.

Write $r_m \dots r_1$ as a product $s_1 \dots s_m$ of simple reflections. The word

(7) $$s_1 \dots s_m s_1 \dots s_m = 1$$

has the cyclic property that $s_i = s_{m+i}$. Assume that the sequence $s_1, ..., s_m$ contains three distinct reflections. Then we can rewrite (7) as

(8) $$s_3 s_1 s_2 \dots s_{m-1} s_3 s_1 s_2 \dots s_{m-1} = 1$$

where $s_3 \neq s_1 \neq s_2$. We will use the following

Lemma 2 (compare with Bourbaki [2], ch. IV, § 1, Lemma 3). *If a word* $s_1 \dots s_n$ *is minimal and the word* $s_1 \dots s_n s$ *is not* $(s \in S)$ *then there is* $1 \leqslant i \leqslant n$ *such that*

(9) $$s_{i+1} \dots s_n s = s_i s_{i+1} \dots s_n$$

*Proof of the Lemma.* Let $1 \leqslant i \leqslant n$ be the maximal index such that $s_{i+1} \dots s_n s$ is minimal and $s_i s_{i+1} \dots s_n s$ is not. Consider the gallery $C_i$, $C_{i+1}, ..., C_n, C$ corresponding by Proposition 1, (iv) to $s_i \dots s_n s$ and let $r_i, ..., r_n, r$ be the corresponding sequence of reflections. Since the gallery $C_i, C_{i+1}, ..., C_n, C$ is not minimal and every subgallery of it is minimal, by Proposition 2, $r_i \neq r_{i+1} \neq \dots \neq r_n$ and $r = r_i$. Thus $s_i s_{i+1} \dots s_n s s_n \dots s_{i+1} s_i = s_i$ which implies $s_{i+1} \dots s_n s = s_i s_{i+1} \dots s_n$ and proves the Lemma.

The word $s_3 s_1 s_2 \dots s_{m-1} s_3$ is not minimal and every subword of it is minimal, therefore, by Lemma 2,

(10) $$s_1 s_2 \dots s_{m-1} s_3 = s_3 s_1 s_2 \dots s_{m-1}$$

that is $s_3$ commutes with $s_1 s_2 \dots s_{m-1}$. This produces two relations

(11) $$s_1 s_2 \dots s_{m-1} s_3 = s_3 s_1 s_2 \dots s_{m-1} = s_{m-1} \dots s_2 s_1 s_3$$

corresponding to three different galleries going from $C_0$ to $C_m$ which contradicts to the assumption that the loop $C_0, ..., C_{2m}$ is perfect.

Thus (7) contains only two reflections $s_1$ and $s_2$, i.e. it has the form

(12) $$(s_1 s_2)^m = 1$$

which is one of the defining relations of the Coxeter group. This completes the proof of the Theorem.