§6. MICRO-DIFFERENTIAL OPERATORS AND THE SYMPLECTIC STRUCTURE ON THE COTANGENT BUNDLE

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **32 (1986)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

§ 5. THE VANISHING CYCLE SHEAF

5.1. Let M be a real manifold and $f: M \to \mathbf{R}$ a continuous map. For a sheaf $\mathscr{F}$ on M, $\mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{F})\,|\,_{f^{-1}(0)}$ is called the (j-th) *vanishing cycle sheaf* of $\mathscr{F}$. Here $\mathbf{R}^+ = \{t \in \mathbf{R}; t \geqslant 0\}$. This measures how the cohomology groups of $\mathscr{F}$ change across the fibers of f. Its algebro-geometric version is studied by Grothendieck-Deligne ([D]).

5.2. Let $(X, \mathscr{O}_X)$ be a complex manifold. Let $f: X \to \mathbf{R}$ be a C^∞-map and consider the vanishing cycle sheaf $\mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{O}_X)\,|\,_{f^{-1}(0)}$. Let s be the section of $f^{-1}(0) \to T^*X$ given by df. Then we have

PROPOSITION 5.2.1 ([KS1] § 3, [K2] § 4.2). *$\mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{O}_X)\,|\,_{f^{-1}(0)}$ has a structure of an $s^{-1}\mathscr{E}_X$-module.*

Let P be a differential operator. If $\sigma(P)$ does not vanish on $s(f^{-1}(0))$, then P has an inverse in $s^{-1}\mathscr{E}_X$ by Proposition 2.2.3. Therefore we obtain

COROLLARY 5.2.2. *If* $\sigma(P)\,|\,_{sf^{-1}(0)} \neq 0$, *then*

$$P: \mathscr{H}^j_{f^{-1}\mathbf{R}^+}(\mathscr{O}_X)\,|\,_{f^{-1}(0)} \to \mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{O}_X)\,|\,_{f^{-1}(0)}$$

is bijective.

5.3. More generally, let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, and set

$$\mathscr{F}^\bullet = \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)\,.$$

Then the preceding corollary shows that

$$\mathbf{R}\Gamma_{f^{-1}\mathbf{R}^+}(\mathscr{F}^\bullet)\,|\,_{f^{-1}(0)} = 0 \quad \text{if} \quad s(f^{-1}(0)) \cap \mathrm{Ch}(\mathscr{M}) = \emptyset\,.$$

Here Ch$\mathscr{M}$ denotes the characteristic variety of $\mathscr{M}$.

5.4. To consider vanishing cycle sheaves is very near to the "microlocal" consideration. In this direction, see [K-S2].

§ 6. MICRO-DIFFERENTIAL OPERATORS AND THE SYMPLECTIC STRUCTURE ON THE COTANGENT BUNDLE

6.1. The ring $\mathscr{E}_X$ is a non-commutative ring. This fact gives rise to new phenomena which are not shared by commutative rings such as the ring of holomorphic functions. They are also closely related to the symplectic structure of the cotangent bundle.

6.2. Let us recall the symplectic structure on the cotangent bundle.

Let θ_X denote the canonical 1-form on the cotangent bundle T^*X of a complex manifold. Then $d\theta_X$ gives the symplectic structure on T^*X. The Hamiltonian map $H: T^*(T^*X) \xrightarrow{\sim} T(T^*X)$ is given by

(6.1.1) $<\eta, \nu> = <d\theta_X, \nu \wedge H(\eta)>$ for $\eta \in T^*(T^*X)$ and $\nu \in T(T^*X)$.

For a function f on T^*X, $H(df)$ is denoted by H_f and the Poisson bracket $\{f, g\}$ is defined as $H_f(g)$. If we denote by $\mathscr{X}$ the Euler vector field (i.e. the infinitesimal action of $\mathbf{C}^*$ on T^*X), then we have

$$\mathscr{X} = H(-\theta_X).$$

With a local coordinate system $(x_1, ..., x_n)$ of X and the associated local coordinate system $(x_1, ..., x_n; \xi_1, ..., \xi_n)$ of T^*X, we have

$$\theta_X = \sum \xi_j dx_j,$$

$$d\theta_X = \sum d\xi_j dx_j,$$

$$H: d\xi_j \mapsto \partial/\partial x_j,\ dx_j \mapsto -\partial/\partial \xi_j$$

$$\mathscr{X} = \sum \xi_j \frac{\partial}{\partial \xi_j},$$

$$\{f, g\} = \sum \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial \xi_j} \frac{\partial f}{\partial x_j} \right).$$

6.3. This structure is deeply related to the ring of micro-differential operators. The first relation between them appears in the following

PROPOSITION 6.3.1. *For* $P \in \mathscr{E}(\lambda)$ *and* $Q \in \mathscr{E}(\mu)$, *set*

$$[P, Q] = PQ - QP \in \mathscr{E}(\lambda + \mu - 1).$$

Then

$$\sigma_{\lambda+\mu-1}([P, Q]) = \{\sigma_\lambda(P), \sigma_\mu(Q)\}.$$

An analytic subset V of T^*X is called *involutive* if $f|_V = g|_V = 0$ implies $\{f, g\}|_V = 0$.

The following theorem exhibits a phenomenon which has no analogue in the commutative case.

THEOREM 6.3.2 ([G]). *Let* $\mathscr{M}$ *be a coherent* $\mathscr{E}_X$*-module defined on an open subset* Ω *of* $\mathring{T}^*X$ *and let* $\mathscr{L}$ *be a* $\mathscr{E}_X(0)|_\Omega$*-module which is a union of coherent* $\mathscr{E}_X(0)$*-modules. Then* $V = \{p \in \Omega;\ \mathscr{L}$ is not coherent over $\mathscr{E}_X(0)$ on any neighborhood of $p\}$ *is an involutive analytic subset of* Ω.

COROLLARY 6.3.3 ([SKK] Chap. II, Theorem 5.3.2, [M]). *For any coherent* $\mathscr{E}_X$*-module* $\mathscr{M}$, Supp $\mathscr{M}$ *is involutive.*

Since any involutive subset has codimension less than or equal to dim X, we have

COROLLARY 6.3.4. *The support of a coherent* $\mathscr{E}_X$*-module has codimension* $\leqslant$ dim X.

After some algebraic calculation, this implies

THEOREM 6.3.5 ([SKK] Chap. II, Theorem 5.3.5). *For any point* $p \in T^*X$, $\mathscr{E}_{X,p}$ *has a global cohomological dimension* dim X.

6.4. An analytic subset Λ of T^*X is called *Lagrangean* if Λ is involutive and dim Λ = dim X. A coherent $\mathscr{E}_X$-module is called *holonomic* if its support is Lagrangean.

§ 7. QUANTIZED CONTACT TRANSFORMATIONS

7.1. In the previous section, we saw that the symplectic structure of T^*X is closely related to micro-differential operators via the relation of commutator and Poisson bracket. In this section, we shall explain another relation.

Definition 7.2.1. Let X and Y be complex manifolds of the same dimension. A morphism φ from an open subset U of T^*X to T^*Y is called a *homogeneous symplectic transformation* if $\varphi^*\theta_Y = \theta_X$.