XI. Continuous operators are not always bounded

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COROLLARY 38. *Assume that the definite space* $(\mathfrak{E}; \langle\ ,\ \rangle)$ *is complete and that the system of types (Corollary 26) is linearly independent in* $\Gamma/2\Gamma$ *(considered as a* $\mathbf{Z}_2$*-vector space) then the conclusions (i), (ii), (iii) of Theorem 37 hold.*

$C(\mathfrak{E})$ in Theorem 37 is not complete (unless finite dimensional). Its quadratic form $\langle\ ,\ \rangle$ can be extended to the completion $\tilde{C}$. By using Theorem 28 one can see that this completion has $L_{\perp\perp}(\tilde{C}) = L_c(\tilde{C})$ if and only if E has $L_{\perp\perp}(E) = L_c(E)$.

XI. CONTINUOUS OPERATORS ARE NOT ALWAYS BOUNDED

XI.1. INTRODUCTION. Let $\mathfrak{E}$ be an infinite dimensional definite space in the sense of Definition 15. A linear map (operator) $h: \mathfrak{E} \to \mathfrak{E}$ is called *bounded* iff there exists $\gamma \in \Gamma$ such that for all $\mathfrak{x} \in \mathfrak{E}$ we have $\varphi\langle h\mathfrak{x}\rangle \geqslant \gamma + \varphi\langle\mathfrak{x}\rangle$.

In [6] A. Fässler gave an explicit example of a continuous operator h on an orthomodular space $\mathfrak{E}$ that is not bounded; she also proved a criterion for boundness which is very useful in the study of the algebra $\mathscr{B}(\mathfrak{E})$ of bounded operators $h: \mathfrak{E} \to \mathfrak{E}$ when $\mathfrak{E}$ is an orthomodular definite space of a certain kind. We shall prove this criterion anew here as its original proof can be shortened considerably.

We shall consider definite spaces that satisfy

(19) $(\mathfrak{E}; \langle\ ,\ \rangle)$ contains a maximal orthogonal family $(\mathfrak{e}_i)_{\mathbf{N}}$ such that the groups $\Theta(\varphi\langle\mathfrak{e}_i\rangle)$ are different.

By (14) we see that (19) is a property of $\mathfrak{E}$, not of $(\mathfrak{e}_i)_{\mathbf{N}}$; Keller's original example of an orthomodular space satisfies (19).

XI.2. FÄSSLER'S CRITERION. In this subsection let $(\mathfrak{E}; \langle\ ,\ \rangle)$ be an infinite dimensional orthomodular space that has (19). Fix a maximal orthogonal family $(\mathfrak{e}_i)_{\mathbf{N}}$ that enjoys (19). If $f: \mathfrak{E} \to \mathfrak{E}$ is given, expand (Lemma 27)

(20) $$f\mathfrak{e}_i = \sum_{j\in\mathbf{N}} \alpha_{ij}\mathfrak{e}_j \qquad (i\in\mathbf{N})$$

THEOREM 39 ([6]). *The linear map* f *is bounded iff it is continuous and satisfies*

(21) $\{\varphi\alpha_{ii} \mid T\,\varphi\langle f\mathfrak{e}_i\rangle = T\,\varphi\langle \mathfrak{e}_i\rangle\}$ *is bounded below.*

The heart of the proof of Theorem 39 is the following consequence of assumption (19).

LEMMA 40 [6]. *If f is continuous then (19) implies that the set* $I := \{i \in \mathbf{N} \mid \varphi\langle f\mathfrak{e}_i\rangle < \varphi\langle \mathfrak{e}_i\rangle \;\&\; \varphi\langle f\mathfrak{e}_i\rangle \not\equiv \varphi\langle \mathfrak{e}_i\rangle \,(\mathrm{mod}\ 2\Gamma)\}$ *is finite.*

Proof. We renumber the $\mathfrak{e}_i$ such that $\Theta(\varphi\langle \mathfrak{e}_i\rangle) \underset{\neq}{\subset} \Theta(\varphi\langle \mathfrak{e}_{i+1}\rangle)$. If we replace $\mathfrak{e}_i$ by a multiple then its group does not change; therefore we may assume without loss of generality that for all $r, s \in \mathbf{N}$ we have

(22) $$r < s \Rightarrow \varphi\langle \mathfrak{e}_r\rangle \in \Theta(\varphi\langle \mathfrak{e}_s\rangle), \quad \varphi\langle \mathfrak{e}_r\rangle \geqslant 0$$

From (22) we obtain that for all $r, s \in \mathbf{N}$

(23) $$r < s \Rightarrow \forall\delta \in \Gamma : \varphi\langle \mathfrak{e}_r\rangle < |\,\varphi\langle \mathfrak{e}_s\rangle + 2\delta\,|$$

If $i \in I$ then $\varphi\langle f\mathfrak{e}_i\rangle \equiv \varphi\langle \mathfrak{e}_j\rangle$ for some $j \neq i$. Let $I_0 \subset I$ be the subset of those i for which the j is smaller than i. Thus, if $i \in I \backslash I_0$ then $\varphi\langle f\mathfrak{e}_i\rangle = \varphi\langle \mathfrak{e}_j\rangle + 2\varphi\alpha_{ij} < \varphi\langle \mathfrak{e}_i\rangle$; so by (23) we must actually have $\varphi\langle f\mathfrak{e}_i\rangle \leqslant -\varphi\langle \mathfrak{e}_i\rangle \leqslant 0$. Since $(\mathfrak{e}_i)$ is a null sequence we see that $I\backslash I_0$ has to be finite (because $\{f\mathfrak{e}_i \mid i \in I\backslash I_0\}$ must also be a null sequence if $I\backslash I_0$ is infinite). Thus, in order to prove Lemma 40 we have to show that I_0 is finite.

The idea in [6] ist to show that for each $i \in I_0$ there is $\lambda_i \in k$ such that $\varphi\langle f(\lambda_i\mathfrak{e}_i)\rangle \leqslant 0$ and $\varphi\langle \lambda_i\mathfrak{e}_i\rangle \geqslant 0$ so that by the same token I_0 must be finite. This is accomplished by choosing, in turn, $\lambda = 1$, $\lambda = \langle f\mathfrak{e}_i\rangle^{-1}$, according as to whether $\varphi\langle f\mathfrak{e}_i\rangle$ is $\leqslant 0$, > 0 respectively.

Proof of Theorem 39. Assume that f is bounded. Continuity is obvious. Let $\gamma \in \Gamma$ be a bound for f and let $\gamma_0 = \min\{0, \gamma\}$. Now $\varphi\langle f\mathfrak{e}_i\rangle = \varphi\langle \alpha_{ii}\mathfrak{e}_i\rangle$ for all i ocurring in (21), i.e., for all $i \in \mathbf{N}\backslash I$ (by assumption (19) we have $T\varphi\langle \mathfrak{e}_i\rangle \neq T\varphi\langle \mathfrak{e}_j\rangle$ for all $i \neq j$). Thus, if $\varphi\alpha_{ii} > 0$ then trivially $\varphi\alpha_{ii} \geqslant \gamma_0$; if $\varphi\alpha_{ii} < 0$ then $\varphi(\alpha_{ii}) > 2\varphi\alpha_{ii} \geqslant \gamma \geqslant \gamma_0$.

Assume conversely that f is continuous and has (21). We show that there is $\gamma_0 \in \Gamma$ with $\varphi\langle f\mathfrak{e}_i\rangle \geqslant \gamma_0 + \varphi\langle \mathfrak{e}_i\rangle$ ($i\in\mathbf{N}$). Let γ be a lower bound for the set in (21) and set $\gamma_0 := \min\{0, 2\gamma, \gamma_1, \ldots, \gamma_n\}$ where $\gamma_\nu := \varphi\langle f\mathfrak{e}_\nu\rangle - \varphi\langle \mathfrak{e}_\nu\rangle$, $\nu \in I$. To finish the proof we conclude $\varphi\langle f\mathfrak{x}\rangle > \varphi\langle \mathfrak{x}\rangle + \gamma_0$ ($\forall\mathfrak{x}$) by continuity of f:

$$\varphi\langle f\sum_1^\infty \xi_i\mathfrak{e}_i\rangle = \varphi\langle f(\xi_{i_0}\mathfrak{e}_{i_0}\rangle) \geqslant \gamma_0 + \varphi\langle \xi_{i_0}\mathfrak{e}_{i_0}\rangle = \gamma_0 + \varphi\langle \mathfrak{x}\rangle .$$